Supplement to "Quantile-Based Nonparametric Inference for First-Price Auctions"

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Abstract

This paper contains supplemental materials for Marmer and Shneyerov (2010). We discuss here how the approach developed in the aforementioned paper can be applied to conducting inference on the optimal reserve price in first-price auctions, report additional simulations results, and provide a detailed proof of the bootstrap result in Marmer and Shneyerov (2010).

S.1 Introduction

This paper contains supplemental materials for Marmer and Shneyerov (2010), MS hereafter. Section S.2 discusses how the approach developed in MS can be applied to conducting inference on the optimal reserve price in first-price auctions. Section S.3 contains the full set of the Monte Carlo simulations results of which only a summary was reported in MS. In Section S.4, we provide a detailed proof of bootstrap Theorem 3 in MS.

The definitions and notation used in this paper are as introduced in MS.

S.2 Inference on the optimal reserve price

In this section, we consider a problem of conducting inference on the optimal reserve price. Several previous articles have studied that problem. Paarsch (1997) develops a parametric approach and applies his estimator to timber auctions in British Columbia. Haile and Tamer (2003) consider the problem of inference in an incomplete model of English auction, derive nonparametric bounds on the reserve price and apply them to the reserve price policy in the US Forest Service auctions. Closer to the subject of our paper, Li, Perrigne, and Vuong (2003) develop a semiparametric method to estimate the optimal reserve price. At a simplified level, their method essentially amounts to re-formulating the problem as a maximum estimator of the seller's expected profit. Strong consistency of the estimator is shown, but its asymptotic distribution is as yet unknown.

We follow Haile and Tamer (2003) and make the following mild technical assumption on the distribution of valuations.[1]

Assumption S.1 *Let c be the seller's own valuation. The function $(p-c)\left(1-F\left(p|x\right)\right)$ is x-a.e. strictly pseudo-concave in p on $\left(\underline{v}\left(x\right), \bar{v}\left(x\right)\right)$.*

Let $r^*\left(x\right)$ denote the optimal reserve price given the covariates value x. Under Assumption S.1 (see the discussion in Haile and Tamer (2003)), $r^*\left(x\right)$ is found as the unique solution to the optimal monopoly pricing problem, and is given by the unique solution to the corresponding first-order condition:

$$r^*\left(x\right)-\frac{1-F\left(r^*\left(x\right)|x\right)}{f\left(r^*\left(x\right)|x\right)}-c=0. \tag{S.1}$$

Remark. Even in the presence of a binding reserve price $r\left(x\right)$ in the data, the optimal reserve price $r^*\left(x\right)$ is still identifiable provided $r^*\left(x\right)>r\left(x\right)$, for the ratio in (S.1) remains the same if we use the truncated distribution $F^*\left(r^*\left(x\right)|x\right)$ defined in Section 5, and the associated density $f^*\left(r^*\left(x\right)|x\right)$, in place of $F\left(r^*\left(x\right)|x\right)$ and $f\left(r^*\left(x\right)|x\right)$. See the discussion of this in Haile and Tamer (2003).

One approach to the inference on $r^*\left(x\right)$ is to estimate it as a solution $\hat{r}^*\left(x\right)$ to (S.1) using consistent estimators for f and F in place of the true unknown functions.

[1] This condition is implied by the standard monotone virtual valuation condition of Myerson (1981). The optimal reserve price result was also obtained in Riley and Samuelson (1981).

However, a difficulty arises because, even though our estimator $\hat{f}(v|x)$ is asymptotically normal, it is not guaranteed to be a continuous function of v. We instead take a direct approach and construct confidens sets (CSs) that do not require a point estimate of $r^*(x)$.

As discussed in Chapter 3.5 of Lehmann and Romano (2005), a natural CS for a parameter can be obtained by inverting a test of a series of simple hypotheses concerning the value of that parameter.[2] We construct CSs for the optimal reserve price by inverting the test of the null hypotheses $H_0(v): r^*(x) = v$. Such hypotheses can be tested by testing the optimal reserve price restriction (S.1) at $r^*(x) = v$. Thus, the CSs are formed by collecting all values v for which the test fails to rejects the null that (S.1) holds at $r^*(x) = v$.

Consider $H_0(v): r^*(x) = v$, and the following test statistic:

$$T(v|x) = \left(Lh^{d+3}\right)^{1/2}\left(v - \frac{1-\hat{F}(v|x)}{\hat{f}(v|x)} - c\right) / \sqrt{\frac{\left(1-\hat{F}(v|x)\right)^2}{\hat{f}^4(v|x)}\hat{V}_f(v,x)},$$

where $\hat{F}$ is defined in (17) in MS, and $\hat{V}_f(v,x)$ is a consistent plug-in estimator of the asymptotic variance of $\hat{f}(v|x)$, see MS Theorem 2. By MS Theorem 2 and Lemma 1(h), $T(r^*(x)|x) \to_d N(0,1)$. Furthermore, due to uniqueness of the solution to (S.1), for any $t > 0$, $P(|T(v|x)| > t | r^*(x) \neq v) \to 1$. A CS for r^* with the asymptotic coverage probability $1-\alpha$ is formed by collecting all v's such that a test based on $T(v|x)$ fails to reject the null at the significance level α:

$$CS_{1-\alpha}(x) = \left\{v \in \hat{\Lambda}(x) : |T(v|x)| \leq z_{1-\alpha/2}\right\},$$

where z_τ is the τ quantile of the standard normal distribution. Asymptotically $CS_{1-\alpha}(x)$ has a correct coverage probability since by construction we have that $P(r^*(x) \in CS_{1-\alpha}(x)) = P\left(|T(r^*(x)|x)| \leq z_{1-\alpha/2}\right) \to 1-\alpha$, provided that $r^*(x) \in \Lambda(x) = [Q(\tau_1|x), Q(\tau_2|x)]$.

When the seller's own evaluation c is unknown, one can treat a CS as a function

[2] CSs obtained by test inversion have been used in the econometrics literature, for example, in the context of instrumental variable regression with weak instruments (Andrews and Stock, 2005), for constructing CSs for the date of a structural break (Elliott and Müller, 2007), and in the case of set identified models (Chernozhukov, Hong, and Tamer, 2007); see also the references on page 1268 of Chernozhukov, Hong, and Tamer (2007).

of c and, using the above approach, construct conditional CSs for chosen values of c.

S.3 Monte Carlo results

In this section, we evaluate the accuracy of the asymptotic normal approximation established in Theorem 2 in MS and that of the bootstrap percentile method discussed in Section 4 in MS. In particular, it is interesting to see whether the boundary effect creates substantial size distortions. We also report here additional simulations results on comparison of our estimator with the estimator of GPV. In addition to the results presented in MS, we also report the results for $v = 0.2, 0.3, 0.7, 0.8$ and $n = 2, 4, 6, 7$. The finite sample performance of the two estimators is compared in terms of bias, mean squared error (MSE), and median absolute deviation. The simulations framework is the same as in Section 6 in MS.

Tables S.1-S.3 report the simulated coverage probabilities for 99%, 95%, and 90% asymptotic confidence intervals (CIs) constructed as

$$\hat{f}(v) \pm z_{1-\alpha/2}\sqrt{\tilde{V}_f(v) / (Lh_2^3)},$$

where $z_{1-\alpha/2}$ denotes the $1-\alpha/2$ quantile of the standard normal distribution, and $\tilde{V}_f(v)$ the second-order corrected estimator of the asymptotic variance of $\hat{f}(v)$ described in Section 3 in MS:

$$\tilde{V}_f(v) = \hat{V}_f(v) + h_2^2\left(\frac{3\hat{f}(v)}{\hat{g}\left(\hat{q}\left(\hat{F}(v)\right)\right)} - \frac{2n\hat{f}^2(v)}{(n-1)\,\hat{g}^2\left(\hat{q}\left(\hat{F}(v)\right)\right)}\right)^2 \hat{V}_{g,0}\left(\hat{q}\left(\hat{F}(v)\right)\right),$$

$$\hat{V}_f(v) = \frac{K_1\hat{F}^2(v)\,\hat{f}^4(v)}{n(n-1)^2\,\hat{g}^5\left(\hat{q}\left(\hat{F}(v)\right)\right)}.$$

In the case of the Uniform $[0,1]$ distribution ($\alpha = 1$, Table S.1), we observe some deviation of the simulated coverage probabilities from the nominal values when the PDF is estimated near the upper boundary and the number of bidders is small ($n = 2, 3$). There is also some deviation of the simulated coverage probabilities from the nominal values for large n and v near the lower boundary of the support. Thus, as one can expect the normal approximation may breakdown near the boundaries of the support. However, away from the boundaries, as the results in Table S.1 indicate, the normal approximation works well and the simulated coverage probabilities are close to their nominal values.

Similar results have been observed in the case of $\alpha = 2$ (Table S.2) and $\alpha = 1/2$ (Table S.3). When $\alpha = 2$, the boundary effect distorting coverage probabilities is somewhat more pronounced near the lower boundary of the support, and less so near the upper boundary. An opposite situation is observed for $\alpha = 1/2$: we see more distortion near the upper boundary and less near the lower boundary of the support. This can be explained by the fact that the PDF is increasing in the case of $\alpha = 2$, so there is relatively more mass near $v = 1$, and it is decreasing when $\alpha = 1/2$, so there is relatively less mass near $v = 0$. We observe good coverage probabilities away from the boundaries.

Tables S.4-S.6 report the coverage probabilities of the percentile bootstrap CIs. The bootstrap percentile confidence intervals are constructed as described in Section 4 in MS. The number of bootstrap samples used to compute $\phi_\tau^\dagger$ in (23) in MS is $M = 199$. The number of Monte Carlo replications used for the bootstrap experiments is 300.[3] When $\alpha = 1$, as reported in Table S.4, for the bootstrap percentile CIs we observe some size distortion only due to the right boundary effect and only for $n = 2$. In all other cases, the bootstrap percentile CIs are found to be very accurate. With a few exceptions, the bootstrap percentile CIs outperform the CIs based on the traditional normal approximation.

Similar results are found for $\alpha = 2$ and $\alpha = 1/2$, see Tables S.5 and S.6. We find that the bootstrap percentile confidence intervals (CIs) have superior accuracy comparing to the CIs based on the traditional normal approximation. Based on these findings, we recommend using the bootstrap percentile method for the inference on the PDF of auction valuations.

We now turn to comparison of our estimator with the GPV's estimator. Table S.7 reports the bias, MSE, and median absolute deviation of the two estimators for $\alpha = 1$. In most cases, the GPV's estimator shows less bias. However neither estimator dominates the other in terms of MSE or median absolute deviation: our quantile-based (QB) estimator appears to be more efficient for small numbers of bidders ($n = 2, 3, 4$), and GPV's is more efficient when $n = 5, 6$, and 7. The GPV's estimator is relatively more efficient when the PDF is upward sloping ($\alpha = 2$) as the results Table S.8

[3] We use a smaller number of replications here because the bootstrap Monte Carlo simulations are significantly more CPU-time consuming.

indicate. However, according to the results in Table S.9, the QB estimator dominates GPV's in the majority of cases when the PDF is downward-sloping ($\alpha = 1/2$).

Tables S.7, S.8, and S.9 also report the average (across replications) standard error for our QB estimator. The variance of the estimator increases with v, since it depends on $F(v)$. This fact is also reflected in the MSE values that increase with v. Interestingly, one can see the same pattern for the MSE of the GPV estimator, which suggests that the GPV variance depends on v as well.

S.4 Proof of bootstrap Theorem 3 in MS

To simplify the notion, we will suppress the subscript indicating the bootstrap sample number for bootstrap objects (m). The bootstrap analogues of the original sample statistics are denoted by the superscript †.

We use $\Phi(\cdot)$ to denote the standard normal CDF. Let $P^\dagger$ denote probability conditional on the original sample $\{(b_{1l}, \ldots, b_{n_l l}, n_l, x_l) : l = 1, \ldots, L\}$. We say $\zeta_L = o_p^\dagger(\lambda_L)$ if $P^\dagger(|\zeta_L/\lambda_L| > \varepsilon) \to_p 0$ for all $\varepsilon > 0$ as $L \to \infty$. We say $\zeta_L = O_p^\dagger(\lambda_L)$ if for all $\varepsilon > 0$ there is $\Delta_\varepsilon > 0$ such that for all $L \geq L_\varepsilon$, $P(P^\dagger(|\zeta_L/\lambda_L| \geq \Delta_\varepsilon) > \varepsilon) < \varepsilon$. We use $E^\dagger$ and $Var^\dagger$ to denote expectation and variance under $P^\dagger$ respectively. Let $\pi^\dagger$ denote the distribution of $n_l^\dagger$ implied by $P^\dagger$, i.e. $\pi^\dagger(n) = P^\dagger(n_l^\dagger = n) = L^{-1}\sum_{l=1}^{L} 1(n_l = n) = \hat{\pi}(n)$, where $\pi(n) = P(n_l = n)$. Lastly, for two CDFs H^1 and H^2, let $d_\infty(H^1, H^2)$ denote the sup-norm distance between H^1 and H^2:

$$d_\infty(H^1, H^2) = \sup_{u \in \mathbb{R}} \left| H^1(u) - H^2(u) \right|.$$

Our proof uses the following two simple lemmas concerning the stochastic order (with respect to $P^\dagger$) of the bootstrap statistics. Let $\hat{\theta}_L$ be a statistic computed using the data in the original sample, and let $\hat{\theta}_L^\dagger$ be the bootstrap analogue of $\hat{\theta}_L$.

Lemma S.1 **(a)** *Suppose that* $\hat{\theta}_L = \theta + o_p(\delta_L)$ *and* $\hat{\theta}_L^\dagger = \hat{\theta}_L + o_p^\dagger(\delta_L)$. *Then,* $\hat{\theta}_L^\dagger = \theta + o_p^\dagger(\delta_L)$.

(b) *Suppose that* $\hat{\theta}_L = \theta + O_p(\delta_L)$ *and* $\hat{\theta}_L^\dagger = \hat{\theta}_L + O_p^\dagger(\delta_L)$. *Then,* $\hat{\theta}_L^\dagger = \theta + O_p^\dagger(\delta_L)$.

Proof of Lemma S.1. For part (a), since $\hat{\theta}_L$ is not random under $P^\dagger$,

$$P^\dagger\left(\delta_L^{-1}\left|\hat{\theta}_L^\dagger - \theta\right| > \varepsilon\right) \leq P^\dagger\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) + P^\dagger\left(\delta_L^{-1}\left|\hat{\theta}_L^\dagger - \hat{\theta}_L\right| > \frac{\varepsilon}{2}\right)$$

$$= 1\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) + o_p(1).$$

For the first summand, we have that for all $\varepsilon, \eta > 0$,

$$P\left(1\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) > \eta\right) = P\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) \to 0.$$

The proof of part (b) is similar. ■

Lemma S.2 *Suppose that* $E^\dagger(\hat{\theta}_L^\dagger)^2 = O_p(\lambda_L^2)$, *then* $\hat{\theta}_L^\dagger = O_p^\dagger(\lambda_L)$.

Proof of Lemma S.2. Since $E^\dagger(\hat{\theta}_L^\dagger)^2 = O_p(\lambda_L^2)$, for all $\varepsilon > 0$ there is $\Delta_\varepsilon > 0$ such that $P(E^\dagger(\hat{\theta}_L^\dagger)^2 > \Delta_\varepsilon^2\lambda_L^2) < \varepsilon$. Let $\tilde{\Delta}_\varepsilon^2 = \Delta_\varepsilon^2/\varepsilon$. Then, we can write

$$P(E^\dagger(\hat{\theta}_L^\dagger)^2 > \tilde{\Delta}_\varepsilon^2\varepsilon\lambda_L^2) < \varepsilon \tag{S.2}$$

for all L large enough. By Markov's inequality,

$$P^\dagger\left(\left|\frac{\hat{\theta}_L^\dagger}{\lambda_L}\right| \geq \tilde{\Delta}_\varepsilon\right) \leq \frac{E^\dagger\left(\hat{\theta}_L^\dagger\right)^2}{\lambda_L^2\tilde{\Delta}_\varepsilon^2}.$$

Thus, for all $\varepsilon > 0$ there is $\tilde{\Delta}_\varepsilon$, such that for all L large enough,

$$P\left(P^\dagger\left(\left|\frac{\hat{\theta}_L^\dagger}{\lambda_L}\right| \geq \tilde{\Delta}_\varepsilon\right) > \varepsilon\right) \leq P\left(\frac{E^\dagger\left(\hat{\theta}_L^\dagger\right)^2}{\tilde{\Delta}_\varepsilon^2\lambda_L^2} > \varepsilon\right) < \varepsilon,$$

where the last inequality is by (S.2). ■

Define

$$H_{g,L}^\dagger(u) = P^\dagger\left(\left(Lh^{d+3}\right)^{1/2}\left(\hat{g}^{\dagger(1)}(b|n,x) - \hat{g}^{(1)}(b|n,x)\right) \leq u\right),$$

Note that $H_{g,L}^\dagger(u)$ depends on x and b. We have the following result.

Lemma S.3 *Let* $[b_1(n,x), b_2(n,x)]$ *be as in (19) in MS. Suppose that Assumptions 1, 2, and 3 with* $k = 1$ *hold. Then, for all* $b \in [b_1(n,x), b_2(n,x)]$, $x \in Interior(\mathcal{X})$ *and* $n \in \mathcal{N}$, $d_\infty(H_{g,L}^\dagger(u), \Phi(u/V_{g,1}^{1/2}(b,n,x))) \to_p 0$.

Proof of Lemma S.3. The result of the lemma follows from Theorem 1 in Mammen (1992) since: (i) $\hat{g}^{(1)}(b|n,x)$ is a linear estimator; (ii) by Lemma 2(a) in MS, $(Lh^{d+3})^{1/2}(\hat{g}^{(1)}(b|n,x) - g^{(1)}(b|n,x)) \to_d N(0, V_{g,1}(b,n,x))$; (iii) d_∞ is a metric; and (iv) due to the under smoothing condition in Assumption 3. ■

Next, by the results in MS Lemma 1, Lemma S.1, and Lemma S.4 below, we have that for $x \in Interior(\mathcal{X})$, $n \in \mathcal{N}$, and $v \in \hat{\Lambda}(x)$,

$$\hat{f}^{\dagger}(v|n,x) - \hat{f}(v|x) = \frac{F(v|x) f^2(v|n,x)}{(n-1) g^3 (q(F(v|x)|n,x)|n,x)} \times \left(\hat{g}^{\dagger(1)}(q(F(v|x)|n,x)) - \hat{g}^{(1)}(q(F(v|x)|n,x))\right) + o_p^{\dagger}\left(Lh^{d+3}\right)^{-1/2}. \quad \text{(S.3)}$$

Note that by Lemma S.3 and (S.3),

$$H_{f,L}^{\dagger}(u) \to_p \Phi\left(\frac{u}{V_f^{1/2}(v,n,x)}\right),$$

where $V_f(v,n,x)$ is defined in Theorem 2 in MS. Furthermore, by Pólya's Theorem, the convergence is uniform in u. The result of the theorem for $H_{f,L}^{\dagger}$ then follows by the triangular inequality $d_\infty(H_{f,L}, H_{f,L}^{\dagger}) \le d_\infty(H_{f,L}, \Phi) + d_\infty(H_{f,L}^{\dagger}, \Phi) \to_p 0$.

Lemma S.4 *Suppose that MS Assumptions 1, 2, and 3 with $k = 1$ hold. Then, for all $x \in Interior(\mathcal{X})$ and $n \in \mathcal{N}$,*

(a) $\hat{\pi}^{\dagger}(n|x) = \hat{\pi}(n|x) + O_p^{\dagger}\left(Lh^d\right)^{-1/2}$.

(b) $\hat{\varphi}^{\dagger}(x) = \hat{\varphi}(x) + O_p^{\dagger}\left(Lh^d\right)^{-1/2}$.

(c) $\sup_{b \in [\underline{b}(n,x), \bar{b}(n,x)]} |\hat{G}^{\dagger}(b|n,x) - \hat{G}(b|n,x)| = O_p^{\dagger}\left(\frac{Lh^d}{\log L}\right)^{-1/2}$.

(d) $\sup_{\tau \in [\varepsilon, 1-\varepsilon]} |\hat{q}^{\dagger}(\tau|n,x) - q(\tau|n,x)| = O_p^{\dagger}\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$, *for all* $0 < \varepsilon < 1/2$.

(e) $\sup_{\tau \in [0,1]} (\lim_{t \downarrow \tau} \hat{q}^{\dagger}(t|n,x) - \hat{q}^{\dagger}(\tau|n,x)) = O_p^{\dagger}\left(\frac{Lh^d}{\log(Lh^d)}\right)^{-1}$.

(f) $\sup_{b \in [b_1(n,x), b_2(n,x)]} |\hat{g}^{(k)\dagger}(b|n,x) - \hat{g}^{(k)}(b|n,x)| = O_p^{\dagger}\left(\frac{Lh^{d+1+2k}}{\log L}\right)^{-1/2}$, $k = 0, \ldots, R$.

(g) $\sup_{\tau \in [\tau_1 - \varepsilon, \tau_2 + \varepsilon]} |\hat{Q}^{\dagger}(\tau|n,x) - Q(\tau|x)| = O_p^{\dagger}\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2} + h^R\right)$, *for some* $\varepsilon > 0$ *such that* $\tau_1 - \varepsilon > 0$ *and* $\tau_2 + \varepsilon < 1$.

(h) $\sup_{v\in\hat{\Lambda}(x)}|\hat{F}^{\dagger}(v|n,x)-F(v|x)|=O_p^{\dagger}\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right)$.

Proof of Lemma S.4. We prove part (b) first. Since $(Lh^d)^{1/2}(\hat{\varphi}(x)-E\varphi(x))$ is asymptotically normal by a standard result for kernel density estimators, by Theorem 1 in Mammen (1992), $(Lh^d)^{1/2}\left(\hat{\varphi}^{\dagger}(x)-\hat{\varphi}(x)\right)=O_p^{\dagger}(1)$. The result in part (b) follows.

For part (a), write

$$\hat{\pi}(n|x)=\hat{\pi}(n,x)\,\hat{\varphi}(x)\,,\text{ where}$$
$$\hat{\pi}(n,x)=\frac{1}{L}\sum_{l=1}^{L}1(n_l=n)\,K_h(x-x_l)\,.$$

By the same argument as in the proof of part (b), $(Lh^d)^{1/2}\left(\hat{\pi}^{\dagger}(n,x)-\hat{\pi}(n,x)\right)$ is asymptotically normal. By the Taylor expansion of $\hat{\pi}^{\dagger}(n|x)$, the result in part (b), and since $\hat{\varphi}(x)$ is bounded away from zero with probability approaching one by Assumption 1(b),

$$\begin{aligned}\left(Lh^d\right)^{1/2}\left(\hat{\pi}^{\dagger}(n|x)-\hat{\pi}(n|x)\right)=&\frac{1}{\hat{\varphi}(x)}\left(Lh^d\right)^{1/2}\left(\hat{\pi}^{\dagger}(n,x)-\hat{\pi}(n,x)\right)\\&-\frac{\hat{\pi}^{\dagger}(n,x)}{\left(\hat{\varphi}(x)\right)^2}\left(Lh^d\right)^{1/2}\left(\hat{\varphi}^{\dagger}(x)-\hat{\varphi}(x)\right)\\&+o\left(\left(Lh^d\right)^{1/2}\left(\hat{\varphi}^{\dagger}(x)-\hat{\varphi}(x)\right)\right)\\=&O_p^{\dagger}(1)\,.\end{aligned}$$

We prove part (c) next. The proof is based on the proof of Lemma B.1 in Newey (1994). For fixed $x\in Interior(\mathcal{X})$ and $n\in\mathcal{N}$, write

$$\begin{aligned}\hat{G}(bn,x)&=\hat{G}(b|n,x)\,\hat{\pi}(n|x)\,\hat{\varphi}(x)\,,\text{ so that}\\\hat{G}(b,n,x)&=\frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}T_{il},\text{ with}\\T_{il}&=1(b_{il}\leq b)\,1(n_l=n)\,K_{*h}(x_l-x)\,,\end{aligned}\tag{S.4}$$

and let

$$\hat{G}^{\dagger}(b,n,x) = \frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l} T_{il}^{\dagger}(b),$$
$$T_{il}^{\dagger}(b) = 1\left(b_{il}^{\dagger} \leq b\right)1\left(n_l^{\dagger} = n\right)K_{*h}\left(x_l^{\dagger} - x\right).$$

Next, for chosen n and x, let

$$I = \left[\underline{b}(n,x), \bar{b}(n,x)\right],$$
$$I = \cup_{k=1}^{J_L} I_k,$$

where the sub-intervals I_k's are non-overlapping and of length

$$s_L = \frac{\log L}{L}. \tag{S.5}$$

Denote as c_k the center of I_k. Note that I, I_k, c_k depend on n and x. Denote as $\kappa(b)$ the interval containing b. Since

$$\hat{G}(b,n,x) = E^{\dagger}T_{il}^{\dagger}(b),$$

we can write

$$\hat{G}^{\dagger}(b,n,x) - \hat{G}(b,n,x) = A_L^{\dagger}(b) - B_L^{\dagger}(b) + C_L^{\dagger}(b), \text{ where}$$
$$A_L^{\dagger}(b) = \frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\left(T_{il}^{\dagger}(b) - T_{il}^{\dagger}\left(c_{\kappa(b)}\right)\right),$$
$$B_L^{\dagger}(b) = \frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\left(E^{\dagger}T_{il}^{\dagger}(b) - E^{\dagger}T_{il}^{\dagger}\left(c_{\kappa(b)}\right)\right),$$
$$C_L^{\dagger}(b) = \frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\left(T_{il}^{\dagger}\left(c_{\kappa(b)}\right) - E^{\dagger}T_{il}^{\dagger}\left(c_{\kappa(b)}\right)\right).$$

In the above decomposition, $A_L^{\dagger}(b)$ is the average of the deviations of $T_{il}^{\dagger}(b)$ from its value computed using the center of the interval containing b, and $B_L^{\dagger}(b)$ is the expected value under $P^{\dagger}$ of $A_L^{\dagger}(b)$. The terms $\sup_{b\in I}|A_L^{\dagger}(b)|$ and $\sup_{b\in I}|B_L^{\dagger}(b)|$ are small when s_L is small.

For $A_L^\dagger$ we have

$$
\begin{aligned}
&\left|T_{il}^\dagger(b) - T_{il}^\dagger\left(c_{\kappa(b)}\right)\right| \\
&\leq h^{-d}(\sup K)^d\, 1\left(n_l^\dagger = n\right)\left|1\left(b_{il}^\dagger \leq b\right) - 1\left(b_{il}^\dagger \leq c_{\kappa(b)}\right)\right| \\
&\leq h^{-d}(\sup K)^d\, 1\left(n_l^\dagger = n\right) 1\left(b_{il}^\dagger \in I_{\kappa(b)}\right),
\end{aligned}
\tag{S.6}
$$

and therefore,

$$
\left|A_L^\dagger(b)\right| \leq h^{-d}(\sup K)^d \frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n} 1\left(n_l^\dagger = n\right) 1\left(b_{il}^\dagger \in I_{\kappa(b)}\right). \tag{S.7}
$$

Next,

$$
\begin{aligned}
&E^\dagger\left(\frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n} 1\left(n_l^\dagger = n\right) 1\left(b_{il}^\dagger \in I_k\right) - P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right)\pi^\dagger(n)\right)^2 \leq \\
&\qquad\leq \frac{P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right)\pi^\dagger(n)}{nL},
\end{aligned}
\tag{S.8}
$$

and by Lemma S.2,

$$
\begin{aligned}
&\frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l} 1\left(n_l^\dagger = n\right) 1\left(b_{il}^\dagger \in I_k\right) \\
&= P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right)\pi^\dagger(n) + O_p^\dagger\left(\frac{P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right)\pi^\dagger(n)}{L}\right)^{1/2} \\
&= P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right)\pi^\dagger(n)\left(1 + O_p^\dagger\left(\frac{1}{P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right)\pi^\dagger(n) L}\right)^{1/2}\right).
\end{aligned}
\tag{S.9}
$$

Now, by a similar argument,

$$
\begin{aligned}
&P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right)\pi^\dagger(n) \\
&= \frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l} 1\left(n_l = n\right) 1\left(b_{il} \in I_k\right)
\end{aligned}
$$

$$
\begin{aligned}
&= P\left(b_{il} \in I_k | n_l = n\right) \pi\left(n\right) \left(1 + O_p\left(\frac{1}{P\left(b_{il} \in I_k | n_l = n\right) \pi\left(n\right) L}\right)^{1/2}\right) \\
&\leq \sup_{k=1,\ldots,J_L} P\left(b_{il} \in I_k | n_l = n\right) \pi\left(n\right) \\
&\quad \times \left(1 + O_p\left(\frac{1}{\inf_{k=1,\ldots,J_L} P\left(b_{il} \in I_k | n_l = n\right) \pi\left(n\right) L}\right)^{1/2}\right). \qquad \text{(S.10)}
\end{aligned}
$$

Furthermore, for all I_k's

$$
\left(\inf_{b \in I, x \in \mathcal{X}} g\left(b | n, x\right)\right) s_L \leq P\left(b_{il} \in I_k | n_l = n\right) \leq \left(\sup_{b \in I, x \in \mathcal{X}} g\left(b | n, x\right)\right) s_L. \qquad \text{(S.11)}
$$

Equations (S.7)-(S.11) together imply that

$$
\begin{aligned}
\left|\sup_{b \in I} A_L^\dagger\left(b\right)\right| &= O_p^\dagger\left(h^{-d} s_L \left(1 + O_p\left(\frac{1}{s_L L}\right)^{1/2}\right)\right) \\
&= O_p^\dagger\left(\frac{\log L}{L h^d}\right), \qquad \text{(S.12)}
\end{aligned}
$$

where the last equality is by (S.5).

By (S.6), (S.10), and (S.11), for $B_L^\dagger\left(b\right)$ we have

$$
\begin{aligned}
\left|\sup_{b \in I} B_L^\dagger\left(b\right)\right| &\leq \sup_{b \in I} E^\dagger \left|T_{il}^\dagger\left(b\right) - T_{il}^\dagger\left(c_{\kappa(b)}\right)\right| \\
&\leq h^{-d} \left(\sup K\right)^d \pi^\dagger\left(n\right) \sup_{k=1,\ldots,J_L} P^\dagger\left(b_{il}^\dagger \in I_k | n_l^\dagger = n\right) \\
&= O_p^\dagger\left(\frac{\log L}{L h^d}\right). \qquad \text{(S.13)}
\end{aligned}
$$

Note that $C_L^\dagger(b)$ depends on b only through c_k's, and therefore

$$
\sup_{b \in I} |C_L^\dagger(b)| \leq \max_{k=1,\ldots,J_L} |C_L^\dagger(c_k)|. \qquad \text{(S.14)}
$$

A Bonferroni inequality implies that for any $\Delta > 0$,

$$
P^\dagger\left(\left(\frac{L h^d}{\log L}\right)^{1/2} \max_{k=1,\ldots,J_L} |C_L^\dagger(c_k)| > \Delta\right) \leq
$$

$$\leq \sum_{k=1}^{J_L} P^{\dagger}\left(\left|\sum_{l=1}^{L}\sum_{i=1}^{n_l}\left(T_{il}^{\dagger}\left(c_k\right)-E^{\dagger}T_{il}^{\dagger}\left(c_k\right)\right)\right|>\Delta nL\left(\frac{\log L}{Lh^d}\right)^{1/2}\right). \quad \text{(S.15)}$$

By (S.4), $|T_{il}^{\dagger}(c_k)| \leq h^{-d}(\sup K)^d$ and

$$\left|T_{il}^{\dagger}\left(c_k\right)-E^{\dagger}T_{il}^{\dagger}\left(c_k\right)\right| \leq 2(\sup K)^d h^{-d}.$$

Further, by (S.8)-(S.11), there is a constant $0 < D_1 < \infty$ such that

$$\begin{aligned} Var^{\dagger}\left(T_{il}^{\dagger}\left(c_k\right)\right) &\leq D_1 h^{-2d} s_L\left(1+o_p\left(1\right)\right) \\ &= D_1 h^{-d}\left(\log L/\left(Lh^d\right)\right)\left(1+o_p\left(1\right)\right). \end{aligned}$$

We therefore can apply Bernstein's inequality (Pollard, 1984, page 193) to obtain

$$\begin{aligned} &P^{\dagger}\left(\left|\sum_{l=1}^{L}\sum_{i=1}^{n_l}\left(T_{il}^{\dagger}\left(c_k\right)-E^{\dagger}T_{il}^{\dagger}\left(c_k\right)\right)\right|>\Delta nL\left(\frac{\log L}{Lh^d}\right)^{1/2}\right) \\ &\leq 2\exp\left(-\frac{1}{2}\frac{\Delta^2 n^2 L^2 \frac{\log L}{Lh^d}}{nLD_1 h^{-d}\left(1+o_p\left(1\right)\right)\frac{\log L}{Lh^d}+\left(2/3\right)\Delta n(\sup K)^d h^{-d} L\left(\frac{\log L}{Lh^d}\right)^{1/2}}\right) \\ &= 2\exp\left(-\frac{1}{2}\frac{\Delta^2 n\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}}{D_1\left(\log L/\left(Lh^d\right)\right)^{1/2}\left(1+o_p\left(1\right)\right)+\left(2/3\right)\Delta(\sup K)^d}\right) \\ &= 2\exp\left(-\frac{\Delta n}{\left(4/3\right)\left(\sup K\right)^d+o_p\left(1\right)}\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}\right), \end{aligned} \quad \text{(S.16)}$$

where the equality in the last line is due to $Lh^d/\log L \to \infty$. The inequalities in (S.14)-(S.16) together with (S.5) imply that there is a constant $0 < D_2 < \infty$ such that

$$\begin{aligned} &P^{\dagger}\left(\left(\frac{Lh^d}{\log L}\right)^{1/2}\sup_{b\in I}|C_L^{\dagger}(b)|>\Delta\right) \\ &\leq 2J_L\exp\left(-\frac{\Delta n}{\left(4/3\right)\left(\sup K\right)^d+o_p\left(1\right)}\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}\right) \\ &\leq D_2 s_L^{-1}\exp\left(-\frac{\Delta n}{\left(4/3\right)\left(\sup K\right)^d+o_p\left(1\right)}\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}\right) \\ &\leq D_2\exp\left(\log L\left(1-\frac{\Delta n}{\left(4/3\right)\left(\sup K\right)^d+o_p\left(1\right)}\left(\frac{Lh^d}{\log L}\right)^{1/2}\right)\right) \end{aligned}$$

$$= o_p(1),$$

where the equality in the last line is by $Lh^d/\log L \to \infty$. By a similar argument as in the proof of Lemma S.2,

$$\sup_{b \in I} |C_L^\dagger(b)| = o_p^\dagger \left(\frac{Lh^d}{\log L} \right)^{-1/2}. \tag{S.17}$$

The result of part (c) follows from (S.12), (S.13), and (S.17).

The proof of part (d) is similar to that of Lemma 1(d) in MS. First, by similar arguments as in the proof of Lemma 1(d), one can show that $\underline{b}(n,x) \le \hat{q}^\dagger(\varepsilon|n,x) \le \hat{q}^\dagger(1-\varepsilon|n,x) \le \bar{b}(n,x)$ with probability $P^\dagger$ approaching one (in probability). Second, one can show that uniformly over $\tau \in [\varepsilon, 1-\varepsilon]$,

$$\hat{G}^\dagger \left(\hat{q}^\dagger(\tau|n,x)|n,x \right) = \tau + O_p^\dagger \left(Lh^d \right)^{-1}$$

Lastly,

$$\begin{aligned}
&G\left(\hat{q}^\dagger(\tau|n,x)|n,x\right) - \hat{G}^\dagger\left(\hat{q}^\dagger(\tau|n,x)|n,x\right) \\
&= G\left(\hat{q}^\dagger(\tau|n,x)|n,x\right) - \tau + O_p^\dagger\left(Lh^d\right)^{-1} \\
&= G\left(\hat{q}^\dagger(\tau|n,x)|n,x\right) - G\left(q\left(\tau|n,x\right)|n,x\right) + O_p^\dagger\left(Lh^d\right)^{-1} \\
&= g\left(\tilde{q}^\dagger\left(\tau|n,x\right)|n,x\right)\left(\hat{q}^\dagger(\tau|n,x) - q\left(\tau|n,x\right)\right) + O_p^\dagger\left(Lh^d\right)^{-1},
\end{aligned}$$

where $\tilde{q}^\dagger$ denotes the mean value, or

$$\begin{aligned}
&\hat{q}^\dagger(\tau|n,x) - q\left(\tau|n,x\right) \\
&= \frac{G\left(\hat{q}^\dagger(\tau|n,x)|n,x\right) - \hat{G}^\dagger\left(\hat{q}^\dagger(\tau|n,x)|n,x\right)}{g\left(\tilde{q}^\dagger\left(\tau|n,x\right)|n,x\right)} + O_p^\dagger\left(Lh^d\right)^{-1} \\
&= \frac{G\left(\hat{q}^\dagger(\tau|n,x)|n,x\right) - \hat{G}\left(\hat{q}^\dagger(\tau|n,x)|n,x\right)}{g\left(\tilde{q}^\dagger\left(\tau|n,x\right)|n,x\right)} \\
&\quad + \frac{\hat{G}\left(\hat{q}^\dagger(\tau|n,x)|n,x\right) - \hat{G}^\dagger\left(\hat{q}^\dagger(\tau|n,x)|n,x\right)}{g\left(\tilde{q}^\dagger\left(\tau|n,x\right)|n,x\right)} + O_p^\dagger\left(Lh^d\right)^{-1},
\end{aligned}$$

and the desired result follows.

The proof of part (e) is similar to that of Lemma 1(e). The proof of part (f) is similar to the proof of part (c) and relies on the fact that, according to Assumption

2 in MS, the derivatives of K are Lipschitz. The proof of parts (g) and (h) is similar to that of Lemma 1(g) and (h). ■

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Table S.1: Simulated coverage probabilities of the **normal approximation** CIs for the PDF of valuations for different points of density estimation (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{1}$ (Uniform [0,1] distribution)

	v						
confidence level	0.2	0.3	0.4	0.5	0.6	0.7	0.8
			$n = 2$				
0.99	0.982	0.975	0.965	0.951	0.909	0.914	0.883
0.95	0.947	0.937	0.926	0.898	0.835	0.838	0.791
0.90	0.882	0.891	0.881	0.860	0.805	0.782	0.754
			$n = 3$				
0.99	0.983	0.984	0.983	0.970	0.949	0.948	0.936
0.95	0.936	0.944	0.948	0.932	0.894	0.896	0.876
0.90	0.869	0.895	0.902	0.893	0.847	0.851	0.820
			$n = 4$				
0.99	0.975	0.982	0.990	0.978	0.966	0.960	0.956
0.95	0.922	0.945	0.956	0.940	0.912	0.919	0.910
0.90	0.851	0.885	0.894	0.893	0.874	0.881	0.867
			$n = 5$				
0.99	0.972	0.977	0.987	0.982	0.974	0.967	0.966
0.95	0.911	0.937	0.949	0.941	0.921	0.932	0.919
0.90	0.842	0.878	0.888	0.888	0.882	0.883	0.885
			$n = 6$				
0.99	0.969	0.976	0.987	0.981	0.976	0.973	0.978
0.95	0.898	0.932	0.940	0.937	0.927	0.933	0.925
0.90	0.829	0.877	0.881	0.885	0.881	0.881	0.884
			$n = 7$				
0.99	0.967	0.973	0.989	0.980	0.974	0.975	0.983
0.95	0.893	0.926	0.932	0.929	0.926	0.933	0.931
0.90	0.823	0.875	0.874	0.883	0.878	0.868	0.883

Table S.2: Simulated coverage probabilities of the **normal approximation** CIs for the PDF of valuations for different points of density estimation (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{2}$

	v						
confidence level	0.2	0.3	0.4	0.5	0.6	0.7	0.8
			$n = 2$				
0.99	0.964	0.949	0.965	0.942	0.933	0.943	0.931
0.95	0.911	0.901	0.910	0.877	0.879	0.878	0.857
0.90	0.855	0.860	0.868	0.831	0.843	0.845	0.788
			$n = 3$				
0.99	0.958	0.968	0.980	0.978	0.964	0.969	0.969
0.95	0.897	0.900	0.927	0.916	0.925	0.928	0.931
0.90	0.817	0.850	0.876	0.865	0.883	0.879	0.874
			$n = 4$				
0.99	0.954	0.970	0.973	0.981	0.979	0.977	0.979
0.95	0.881	0.890	0.926	0.927	0.929	0.938	0.939
0.90	0.797	0.830	0.874	0.867	0.880	0.890	0.896
			$n = 5$				
0.99	0.956	0.961	0.971	0.981	0.982	0.981	0.979
0.95	0.868	0.883	0.917	0.930	0.927	0.935	0.935
0.90	0.791	0.820	0.850	0.870	0.865	0.889	0.887
			$n = 6$				
0.99	0.952	0.957	0.970	0.983	0.984	0.983	0.980
0.95	0.861	0.887	0.903	0.918	0.919	0.932	0.936
0.90	0.789	0.813	0.835	0.862	0.853	0.870	0.880
			$n = 7$				
0.99	0.953	0.960	0.975	0.977	0.981	0.979	0.978
0.95	0.859	0.882	0.889	0.915	0.910	0.925	0.932
0.90	0.792	0.810	0.824	0.855	0.845	0.858	0.860

Table S.3: Simulated coverage probabilities of the **normal approximation** CIs for the PDF of valuations for different points of density estimation (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{1/2}$

	v						
confidence level	0.2	0.3	0.4	0.5	0.6	0.7	0.8
			$n = 2$				
0.99	0.976	0.966	0.937	0.899	0.877	0.817	0.780
0.95	0.935	0.915	0.875	0.827	0.794	0.716	0.698
0.90	0.876	0.870	0.818	0.772	0.738	0.656	0.625
			$n = 3$				
0.99	0.983	0.984	0.954	0.926	0.908	0.875	0.849
0.95	0.948	0.933	0.901	0.871	0.853	0.796	0.772
0.90	0.890	0.886	0.861	0.829	0.807	0.735	0.716
			$n = 4$				
0.99	0.984	0.987	0.967	0.951	0.933	0.907	0.880
0.95	0.954	0.946	0.921	0.895	0.883	0.834	0.819
0.90	0.890	0.892	0.878	0.855	0.835	0.792	0.764
			$n = 5$				
0.99	0.985	0.988	0.977	0.963	0.952	0.930	0.908
0.95	0.950	0.949	0.935	0.913	0.900	0.860	0.845
0.90	0.891	0.898	0.884	0.876	0.863	0.823	0.797
			$n = 6$				
0.99	0.984	0.991	0.982	0.966	0.959	0.941	0.932
0.95	0.944	0.950	0.936	0.920	0.913	0.889	0.869
0.90	0.889	0.903	0.886	0.884	0.881	0.839	0.821
			$n = 7$				
0.99	0.982	0.990	0.983	0.973	0.962	0.949	0.943
0.95	0.940	0.951	0.936	0.925	0.925	0.899	0.893
0.90	0.886	0.903	0.884	0.887	0.890	0.861	0.842

Table S.4: Simulated coverage probabilities of the **bootstrap percentile** CIs for PDF of valuations for different points of density estimation (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{1}$ (Uniform [0,1] distribution)

	v						
confidence level	0.2	0.3	0.4	0.5	0.6	0.7	0.8
			$n = 2$				
0.99	0.997	0.980	0.997	0.987	0.990	0.993	0.987
0.95	0.957	0.957	0.953	0.930	0.940	0.937	0.923
0.90	0.890	0.913	0.913	0.887	0.897	0.840	0.827
			$n = 3$				
0.99	1.000	0.993	0.997	0.987	0.987	0.993	0.993
0.95	0.940	0.960	0.957	0.937	0.953	0.957	0.933
0.90	0.890	0.910	0.917	0.887	0.900	0.863	0.880
			$n = 4$				
0.99	1.000	0.990	0.993	0.980	0.987	0.993	0.990
0.95	0.953	0.963	0.963	0.930	0.957	0.957	0.937
0.90	0.870	0.907	0.917	0.887	0.900	0.903	0.890
			$n = 5$				
0.99	0.997	0.990	0.993	0.987	0.987	0.993	0.987
0.95	0.947	0.950	0.963	0.927	0.957	0.960	0.933
0.90	0.873	0.913	0.910	0.873	0.897	0.900	0.893
			$n = 6$				
0.99	0.997	0.993	0.993	0.980	0.990	0.990	0.987
0.95	0.953	0.950	0.967	0.923	0.957	0.947	0.943
0.90	0.883	0.920	0.913	0.870	0.907	0.880	0.887
			$n = 7$				
0.99	0.990	0.990	0.993	0.977	0.993	0.987	0.990
0.95	0.947	0.953	0.963	0.917	0.957	0.950	0.933
0.90	0.883	0.923	0.903	0.863	0.897	0.887	0.883

Table S.5: Simulated coverage probabilities of the **bootstrap percentile** CIs for PDF of valuations for different points of density estimation (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{2}$

	v						
confidence level	0.2	0.3	0.4	0.5	0.6	0.7	0.8
			$n = 2$				
0.99	0.983	0.987	0.980	0.990	0.987	0.987	0.990
0.95	0.943	0.953	0.943	0.953	0.933	0.927	0.927
0.90	0.893	0.903	0.887	0.923	0.877	0.877	0.877
			$n = 3$				
0.99	0.987	0.977	0.983	0.987	0.993	0.993	0.993
0.95	0.950	0.937	0.943	0.957	0.963	0.930	0.940
0.90	0.900	0.897	0.880	0.930	0.917	0.893	0.897
			$n = 4$				
0.99	0.990	0.980	0.980	0.987	0.993	0.993	0.993
0.95	0.937	0.940	0.940	0.953	0.963	0.920	0.947
0.90	0.907	0.903	0.867	0.920	0.907	0.873	0.893
			$n = 5$				
0.99	0.987	0.987	0.990	0.990	0.997	0.997	0.997
0.95	0.950	0.930	0.923	0.953	0.960	0.913	0.953
0.90	0.910	0.900	0.880	0.913	0.917	0.873	0.903
			$n = 6$				
0.99	0.990	0.987	0.987	0.987	0.993	0.990	0.997
0.95	0.953	0.937	0.930	0.953	0.950	0.930	0.950
0.90	0.920	0.900	0.887	0.907	0.917	0.873	0.907
			$n = 7$				
0.99	0.987	0.987	0.987	0.990	0.997	0.990	0.997
0.95	0.947	0.947	0.937	0.953	0.957	0.933	0.957
0.90	0.910	0.883	0.890	0.903	0.903	0.877	0.907

Table S.6: Simulated coverage probabilities of the **bootstrap percentile** CIs for PDF of valuations for different points of density estimation (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{1/2}$

	v						
confidence level	0.2	0.3	0.4	0.5	0.6	0.7	0.8
			$n = 2$				
0.99	0.993	0.993	0.980	0.987	0.980	0.973	0.983
0.95	0.933	0.943	0.930	0.907	0.900	0.883	0.910
0.90	0.870	0.917	0.897	0.813	0.803	0.753	0.803
			$n = 3$				
0.99	0.997	0.993	0.983	0.983	0.977	0.980	0.980
0.95	0.937	0.957	0.943	0.927	0.917	0.913	0.917
0.90	0.890	0.927	0.900	0.843	0.820	0.787	0.840
			$n = 4$				
0.99	0.997	0.987	0.987	0.990	0.980	0.983	0.983
0.95	0.943	0.960	0.953	0.937	0.933	0.927	0.940
0.90	0.893	0.907	0.910	0.863	0.847	0.830	0.843
			$n = 5$				
0.99	0.993	0.987	0.987	0.993	0.983	0.980	0.977
0.95	0.960	0.953	0.963	0.933	0.943	0.950	0.930
0.90	0.900	0.927	0.903	0.873	0.877	0.860	0.873
			$n = 6$				
0.99	0.993	0.987	0.983	0.993	0.987	0.983	0.980
0.95	0.953	0.953	0.960	0.943	0.953	0.943	0.933
0.90	0.900	0.913	0.897	0.873	0.893	0.883	0.887
			$n = 7$				
0.99	0.993	0.987	0.987	0.993	0.983	0.987	0.977
0.95	0.957	0.953	0.957	0.947	0.957	0.957	0.923
0.90	0.913	0.917	0.897	0.873	0.907	0.890	0.900

Table S.7: Bias, MSE and median absolute deviation of the quantile-based (QB) and GPV estimators, and the average standard error (second-order corrected) of the QB estimator, for different points of density estimations (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{1}$ (Uniform [0,1] distribution)

	Bias		MSE		Med abs deviation		
v	QB	GPV	QB	GPV	QB	GPV	Std err QB
				$n = 2$			
0.2	**-0.0025**	0.0030	**0.0126**	0.0218	**0.0909**	0.1186	0.1073
0.3	-0.0191	**-0.0022**	**0.0216**	0.0439	**0.1178**	0.1683	0.1519
0.4	-0.0173	**0.0099**	**0.0405**	0.0768	**0.1556**	0.2189	0.2004
0.5	-0.0270	**0.0227**	**0.0560**	0.1177	**0.1801**	0.2696	0.2471
0.6	-0.0743	**-0.0068**	**0.0764**	0.1571	**0.2123**	0.3141	0.2752
0.7	-0.0722	**0.0195**	**0.1027**	0.2061	**0.2405**	0.3681	0.3312
0.8	-0.0917	**0.0061**	**0.2016**	0.2366	**0.2744**	0.3959	0.4143
				$n = 3$			
0.2	**0.0004**	0.0025	**0.0077**	0.0082	**0.0710**	0.0731	0.0793
0.3	-0.0111	**-0.0035**	**0.0114**	0.0145	**0.0851**	0.0970	0.1073
0.4	-0.0063	**0.0045**	**0.0194**	0.0245	**0.1094**	0.1245	0.1382
0.5	**-0.0056**	0.0147	**0.0284**	0.0371	**0.1299**	0.1522	0.1701
0.6	-0.0342	**-0.0059**	**0.0402**	0.0519	**0.1556**	0.1813	0.1947
0.7	-0.0264	**0.0114**	**0.0503**	0.0720	**0.1781**	0.2161	0.2287
0.8	-0.0433	**0.0017**	**0.0613**	0.0857	**0.1953**	0.2372	0.2578
				$n = 4$			
0.2	**0.0013**	0.0021	0.0059	**0.0050**	0.0619	**0.0567**	0.0667
0.3	-0.0084	**-0.0039**	0.0077	**0.0077**	0.0697	**0.0696**	0.0860
0.4	-0.0031	**0.0023**	**0.0121**	0.0124	**0.0871**	0.0886	0.1079
0.5	**0.0004**	0.0110	**0.0175**	0.0183	**0.1033**	0.1071	0.1311
0.6	-0.0204	**-0.0044**	**0.0248**	0.0256	**0.1226**	0.1275	0.1505
0.7	-0.0115	**0.0082**	**0.0315**	0.0360	**0.1415**	0.1514	0.1764
0.8	-0.0233	**0.0002**	**0.0380**	0.0429	**0.1545**	0.1660	0.1982

Table S.7: Continued ($\alpha = \mathbf{1}$)

	Bias		MSE		Med abs deviation		
v	QB	GPV	QB	GPV	QB	GPV	Std err QB
				$n = 5$			
0.2	**0.0016**	0.0019	0.0050	**0.0037**	0.0570	**0.0490**	0.0600
0.3	-0.0072	**-0.0040**	0.0060	**0.0052**	0.0611	**0.0565**	0.0741
0.4	-0.0017	**0.0013**	0.0087	**0.0078**	0.0744	**0.0703**	0.0905
0.5	**0.0026**	0.0088	0.0124	**0.0113**	0.0877	**0.0843**	0.1083
0.6	-0.0138	**-0.0035**	0.0171	**0.0156**	0.1026	**0.0997**	0.1241
0.7	**-0.0051**	0.0064	0.0220	**0.0217**	0.1182	**0.1170**	0.1444
0.8	-0.0147	**-0.0003**	0.0262	**0.0259**	**0.1278**	0.1284	0.1615
				$n = 6$			
0.2	0.0018	**0.0018**	0.0046	**0.0032**	0.0540	**0.0448**	0.0560
0.3	-0.0065	**-0.0040**	0.0051	**0.0039**	0.0559	**0.0493**	0.0667
0.4	-0.0010	**0.0007**	0.0069	**0.0057**	0.0665	**0.0598**	0.0795
0.5	**0.0037**	0.0074	0.0096	**0.0079**	0.0774	**0.0708**	0.0937
0.6	-0.0101	**-0.0029**	0.0129	**0.0108**	0.0895	**0.0831**	0.1068
0.7	**-0.0020**	0.0053	0.0167	**0.0148**	0.1026	**0.0961**	0.1231
0.8	-0.0100	**-0.0005**	0.0195	**0.0175**	0.1105	**0.1055**	0.1374
				$n = 7$			
0.2	0.0019	**0.0017**	0.0043	**0.0028**	0.0522	**0.0423**	0.0535
0.3	-0.0061	**-0.0040**	0.0045	**0.0033**	0.0526	**0.0449**	0.0618
0.4	-0.0006	**0.0004**	0.0059	**0.0045**	0.0613	**0.0533**	0.0721
0.5	**0.0042**	0.0064	0.0079	**0.0061**	0.0704	**0.0620**	0.0836
0.6	-0.0077	**-0.0024**	0.0103	**0.0082**	0.0805	**0.0723**	0.0947
0.7	**-0.0004**	0.0045	0.0133	**0.0109**	0.0917	**0.0824**	0.1082
0.8	-0.0075	**-0.0005**	0.0152	**0.0128**	0.0977	**0.0903**	0.1202

Table S.8: Bias, MSE and median absolute deviation of the quantile-based (QB) and GPV estimators, and the average standard error (second-order corrected) of the QB estimator, for different points of density estimations (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{2}$

	Bias		MSE		Med abs deviation		
v	QB	GPV	QB	GPV	QB	GPV	Std err QB
				$n = 2$			
0.2	-0.0024	**0.0008**	**0.0043**	0.0048	**0.0508**	0.0555	0.0588
0.3	-0.0153	**-0.0056**	**0.0126**	0.0159	**0.0867**	0.1010	0.1028
0.4	-0.0144	**0.0053**	**0.0268**	0.0337	**0.1257**	0.1465	0.1596
0.5	-0.0380	**-0.0097**	**0.0477**	0.0620	**0.1702**	0.1983	0.2173
0.6	-0.0443	**0.0027**	**0.0727**	0.1015	**0.2129**	0.2588	0.2855
0.7	-0.0562	**0.0197**	**0.1197**	0.1621	**0.2602**	0.3228	0.3617
0.8	-0.0912	**-0.0110**	0.2400	**0.2360**	**0.3379**	0.3920	0.4430
				$n = 3$			
0.2	-0.0013	**0.0003**	0.0022	**0.0019**	0.0377	**0.0346**	0.0391
0.3	-0.0072	**-0.0034**	0.0057	**0.0051**	0.0595	**0.0569**	0.0660
0.4	-0.0037	**0.0028**	0.0113	**0.0106**	0.0837	**0.0817**	0.0995
0.5	-0.0166	**-0.0084**	0.0194	**0.0188**	0.1116	**0.1091**	0.1345
0.6	-0.0137	**0.0029**	0.0310	**0.0299**	**0.1401**	0.1404	0.1779
0.7	**-0.0103**	0.0133	0.0499	**0.0478**	**0.1716**	0.1735	0.2242
0.8	-0.0384	**-0.0052**	**0.0730**	0.0733	**0.2136**	0.2172	0.2656
				$n = 4$			
0.2	-0.0012	**0.0001**	0.0018	**0.0013**	0.0337	**0.0288**	0.0332
0.3	-0.0049	**-0.0024**	0.0039	**0.0029**	0.0494	**0.0431**	0.0523
0.4	**-0.0015**	0.0018	0.0071	**0.0057**	0.0669	**0.0602**	0.0755
0.5	-0.0103	**-0.0066**	0.0113	**0.0095**	0.0858	**0.0779**	0.1007
0.6	-0.0065	**0.0019**	0.0182	**0.0150**	0.1077	**0.0990**	0.1311
0.7	**-0.0015**	0.0099	0.0281	**0.0232**	0.1309	**0.1207**	0.1637
0.8	-0.0186	**-0.0037**	0.0423	**0.0356**	0.1623	**0.1507**	0.1957

Table S.8: Continued ($\alpha = \mathbf{2}$)

	Bias		MSE		Med abs deviation		
v	QB	GPV	QB	GPV	QB	GPV	Std err QB
				$n = 5$			
0.2	-0.0012	**-0.0001**	0.0016	**0.0011**	0.0322	**0.0265**	0.0311
0.3	-0.0039	**-0.0019**	0.0032	**0.0022**	0.0447	**0.0376**	0.0459
0.4	**-0.0008**	0.0014	0.0054	**0.0040**	0.0585	**0.0503**	0.0635
0.5	-0.0075	**-0.0054**	0.0080	**0.0062**	0.0721	**0.0629**	0.0831
0.6	-0.0041	**0.0011**	0.0127	**0.0097**	0.0905	**0.0794**	0.1062
0.7	**0.0012**	0.0079	0.0190	**0.0144**	0.1085	**0.0949**	0.1312
0.8	-0.0120	**-0.0030**	0.0277	**0.0217**	0.1320	**0.1172**	0.1566
				$n = 6$			
0.2	-0.0014	**-0.0002**	0.0016	**0.0011**	0.0315	**0.0255**	0.0302
0.3	-0.0033	**-0.0016**	0.0028	**0.0019**	0.0424	**0.0347**	0.0426
0.4	**-0.0006**	0.0011	0.0046	**0.0032**	0.0538	**0.0451**	0.0569
0.5	-0.0058	**-0.0046**	0.0064	**0.0047**	0.0641	**0.0547**	0.0729
0.6	-0.0030	**0.0006**	0.0100	**0.0072**	0.0800	**0.0683**	0.0914
0.7	**0.0023**	0.0066	0.0144	**0.0103**	0.0947	**0.0804**	0.1115
0.8	-0.0087	**-0.0026**	0.0203	**0.0151**	0.1134	**0.0975**	0.1324
				$n = 7$			
0.2	-0.0014	**-0.0002**	0.0016	**0.0010**	0.0312	**0.0249**	0.0299
0.3	-0.0029	**-0.0014**	0.0026	**0.0017**	0.0411	**0.0331**	0.0407
0.4	**-0.0004**	0.0009	0.0041	**0.0028**	0.0509	**0.0421**	0.0529
0.5	-0.0048	**-0.0040**	0.0055	**0.0039**	0.0591	**0.0497**	0.0664
0.6	-0.0024	**0.0001**	0.0084	**0.0058**	0.0732	**0.0613**	0.0818
0.7	**0.0028**	0.0057	0.0117	**0.0080**	0.0858	**0.0713**	0.0986
0.8	-0.0068	**-0.0023**	0.0161	**0.0115**	0.1011	**0.0848**	0.1163

Table S.9: Bias, MSE and median absolute deviation of the quantile-based (QB) and GPV estimators, and the average standard error (second-order corrected) of the QB estimator, for different points of density estimations (v), numbers of bidders (n) and auctions (L), sample size $nL = 4200$, and the distribution parameter $\alpha = \mathbf{1/2}$

	Bias		MSE		Med abs deviation		
v	QB	GPV	QB	GPV	QB	GPV	Std err QB
				$n = 2$			
0.2	-0.0186	**-0.0102**	**0.0220**	0.0576	**0.1195**	0.1891	0.1497
0.3	-0.0201	**0.0018**	**0.0343**	0.1059	**0.1479**	0.2512	0.1886
0.4	-0.0458	**-0.0190**	**0.0706**	0.1409	**0.1737**	0.2902	0.2269
0.5	-0.0625	**0.0010**	**0.0548**	0.1800	**0.1790**	0.3330	0.2486
0.6	-0.0706	**-0.0137**	0.5800	**0.1700**	**0.2100**	0.3238	0.7302
0.7	-0.1047	**0.0020**	**0.0756**	0.1771	**0.2107**	0.3397	0.2954
0.8	-0.1042	**0.0107**	0.2375	**0.1719**	**0.2342**	0.3332	0.5659
				$n = 3$			
0.2	-0.0124	**-0.0040**	**0.0144**	0.0241	**0.0976**	0.1247	0.1194
0.3	-0.0110	**-0.0009**	**0.0213**	0.0412	**0.1163**	0.1631	0.1463
0.4	-0.0302	**-0.0110**	**0.0299**	0.0572	**0.1353**	0.1892	0.1694
0.5	-0.0323	**0.0030**	**0.0352**	0.0770	**0.1482**	0.2242	0.1963
0.6	-0.0596	**-0.0094**	**0.0393**	0.0781	**0.1518**	0.2214	0.2091
0.7	-0.0763	**0.0053**	0.1213	**0.0948**	**0.1771**	0.2495	0.2785
0.8	-0.0742	**0.0149**	**0.0984**	0.0997	**0.1841**	0.2539	0.2962
				$n = 4$			
0.2	-0.0089	**-0.0006**	**0.0109**	0.0136	**0.0848**	0.0946	0.1017
0.3	-0.0070	**-0.0004**	**0.0146**	0.0219	**0.0969**	0.1193	0.1212
0.4	-0.0199	**-0.0072**	**0.0206**	0.0308	**0.1140**	0.1393	0.1399
0.5	-0.0146	**0.0032**	**0.0278**	0.0418	**0.1287**	0.1653	0.1646
0.6	-0.0393	**-0.0061**	**0.0284**	0.0432	**0.1301**	0.1662	0.1750
0.7	-0.0438	**0.0048**	**0.0469**	0.0565	**0.1466**	0.1927	0.2027
0.8	-0.0530	**0.0128**	**0.0455**	0.0627	**0.1534**	0.2018	0.2164

Table S.9: Continued ($\alpha = \mathbf{1/2}$)

v	Bias QB	Bias GPV	MSE QB	MSE GPV	Med abs deviation QB	Med abs deviation GPV	Std err QB
				$n = 5$			
0.2	-0.0067	**0.0015**	**0.0089**	0.0092	**0.0768**	0.0780	0.0903
0.3	-0.0046	**0.0004**	**0.0110**	0.0137	**0.0842**	0.0946	0.1048
0.4	-0.0142	**-0.0053**	**0.0156**	0.0195	**0.0992**	0.1106	0.1201
0.5	-0.0077	**0.0035**	**0.0208**	0.0261	**0.1130**	0.1304	0.1400
0.6	-0.0278	**-0.0039**	**0.0211**	0.0273	**0.1136**	0.1320	0.1500
0.7	-0.0299	**0.0037**	**0.0292**	0.0366	**0.1277**	0.1549	0.1699
0.8	-0.0363	**0.0102**	**0.0329**	0.0419	**0.1353**	0.1649	0.1838
				$n = 6$			
0.2	-0.0052	**0.0028**	0.0076	**0.0069**	0.0712	**0.0678**	0.0824
0.3	-0.0030	**0.0012**	**0.0087**	0.0096	**0.0753**	0.0792	0.0934
0.4	-0.0107	**-0.0042**	**0.0124**	0.0136	**0.0886**	0.0925	0.1059
0.5	-0.0046	**0.0037**	**0.0162**	0.0180	**0.1005**	0.1079	0.1221
0.6	-0.0206	**-0.0026**	**0.0164**	0.0189	**0.1009**	0.1097	0.1316
0.7	-0.0213	**0.0029**	**0.0216**	0.0255	**0.1142**	0.1291	0.1478
0.8	-0.0257	**0.0084**	**0.0249**	0.0295	**0.1206**	0.1383	0.1601
				$n = 7$			
0.2	-0.0041	**0.0038**	0.0068	**0.0056**	0.0672	**0.0611**	0.0767
0.3	-0.0019	**0.0018**	0.0073	**0.0072**	0.0689	**0.0688**	0.0851
0.4	-0.0086	**-0.0034**	0.0103	**0.0101**	0.0806	**0.0800**	0.0954
0.5	**-0.0029**	0.0037	**0.0131**	0.0132	**0.0907**	0.0925	0.1088
0.6	-0.0159	**-0.0019**	**0.0132**	0.0139	**0.0908**	0.0940	0.1176
0.7	-0.0156	**0.0025**	**0.0171**	0.0188	**0.1027**	0.1106	0.1313
0.8	-0.0185	**0.0072**	**0.0202**	0.0218	**0.1094**	0.1186	0.1427