LECTURE 13
SPURIOUS REGRESSION, TESTING FOR UNIT ROOT

Spurious regression

In this section, we consider the situation when is one unit root process, say Y_t, is regressed against another unit root process, say X_t, while the two processes are unrelated. Assume that

- $X_t = X_{t-1} + u_t$, $X_0 = 0$.
- $Y_t = Y_{t-1} + v_t$, $Y_0 = 0$.
- $\begin{pmatrix} u_t \\ v_t \end{pmatrix} = C(L)\varepsilon_t$.
- $\{\varepsilon_t\}$ is iid, $E\varepsilon_t = 0$, $E\varepsilon_t\varepsilon_t' = \Sigma$ a positive definite matrix:
$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}.$$
- $\sum_{j=1}^{\infty} j^{1/2} \|C_j\| < \infty$, $C(1)$ is non-singular.

We can also set the initial values of X and Y to be different from zero: $X_0 = o_p\left(n^{1/2}\right)$, and $Y_0 = o_p\left(n^{1/2}\right)$.

Let

$$\begin{aligned} \Omega &= C(1)\Sigma C(1)' \\ &= \begin{pmatrix} \omega_u^2 & \omega_{uv} \\ \omega_{uv} & \omega_v^2 \end{pmatrix}, \end{aligned}$$

where ω_u^2 and ω_v^2 are the long-run variances of u_t and v_t respectively, and ω_{uv} is the long-run covariance between u_t and v_t.

Let $\widehat{\beta}_n$ be the OLS regression coefficient:

$$\widehat{\beta}_n = \frac{\sum_{t=1}^{n} X_t Y_t}{\sum_{t=1}^{n} X_t^2}.$$

Here we consider the regression without an intercept, however, essentially the same results can be obtained for the regression with an intercept.

Let

$$\begin{aligned} W_{X,n}(r) &= n^{-1/2} X_{[nr]}, \\ W_{Y,n}(r) &= n^{-1/2} Y_{[nr]}. \end{aligned}$$

From the FCLT we have

$$\begin{pmatrix} W_{X,n}(r) \\ W_{Y,n}(r) \end{pmatrix} \Longrightarrow \begin{pmatrix} B_X(r) \\ B_Y(r) \end{pmatrix} = \Omega'^{1/2} \begin{pmatrix} W_X(r) \\ W_Y(r) \end{pmatrix},$$

where $W_X(r)$ and $W_Y(r)$ are two independent standard Brownian motions. Notice that convergence is joint, which is important for all subsequent results.

Consider first

$$
\begin{aligned}
n^{-3/2}\sum_{t=1}^{n}X_{t-1} &= n^{-1}\sum_{t=1}^{n}W_{X,n}\left(\frac{t-1}{n}\right)\\
&= n^{-1}\left(0+W_{X,n}\left(\frac{1}{n}\right)+W_{X,n}\left(\frac{2}{n}\right)+\ldots+W_{X,n}\left(\frac{n-1}{n}\right)\right).
\end{aligned}
$$

The function $W_{X,n}(r)$ is cadlag, constant on the interval $(t-1)/n \le r < t/n$, and, therefore,

$$
\begin{aligned}
n^{-1}W_{X,n}\left(\frac{t-1}{n}\right) &= \int_{(t-1)/n}^{t/n} dr W_{X,n}\left(\frac{t-1}{n}\right)\\
&= \int_{(t-1)/n}^{t/n} W_{X,n}(r)\,dr,
\end{aligned}
$$

so that

$$
\begin{aligned}
0 &= \int_{0}^{1/n} W_{X,n}(r)\,dr,\\
n^{-1}W_{X,n}\left(\frac{1}{n}\right) &= \int_{1/n}^{2/n} W_{X,n}(r)\,dr,\\
&\cdots\\
n^{-1}W_{X,n}\left(\frac{n-1}{n}\right) &= \int_{(n-1)/n}^{1} W_{X,n}(r)\,dr.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
n^{-3/2}\sum_{t=1}^{n}X_{t-1} &= \sum_{t=1}^{n}\int_{(t-1)/n}^{t/n} W_{X,n}(r)\,dr\\
&= \int_{0}^{1} W_{X,n}(r)\,dr\\
&\to_d \int_{0}^{1} B_X(r)\,dr.
\end{aligned}
$$

where the last result is by the CMT. Next,

$$
\begin{aligned}
n^{-3/2}\sum_{t=1}^{n}X_t &= n^{-3/2}\sum_{t=1}^{n}(X_{t-1}+u_t)\\
&= n^{-3/2}\sum_{t=1}^{n}X_{t-1}+n^{-3/2}\sum_{t=1}^{n}u_t\\
&= n^{-3/2}\sum_{t=1}^{n}X_{t-1}+o_p(1)\\
&\to_d \int_{0}^{1} B_X(r)\,dr.
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
n^{-2}\sum_{t=1}^{n}X_t^2 &= n^{-2}\sum_{t=1}^{n}(X_{t-1}+u_t)^2\\
&= n^{-2}\sum_{t=1}^{n}X_{t-1}^2+2n^{-2}\sum_{t=1}^{n}X_{t-1}u_t+n^{-2}\sum_{t=1}^{n}u_t^2.\\
&= n^{-2}\sum_{t=1}^{n}X_{t-1}^2+2n^{-2}\sum_{t=1}^{n}X_{t-1}u_t+o_p(1).
\end{aligned}
$$

First,

$$n^{-2}\sum_{t=1}^{n}X_{t-1}^{2}=n^{-1}\sum_{t=1}^{n}\left(n^{-1/2}X_{t-1}\right)^{2}$$
$$=\int_{0}^{1}W_{X,n}^{2}\left(r\right)dr$$
$$\rightarrow_{d}\int_{0}^{1}B_{X}^{2}\left(r\right)dr.$$

Next,

$$2X_{t-1}u_{t}=X_{t}^{2}-X_{t-1}^{2}-u_{t}^{2},$$

and

$$\begin{aligned}n^{-1}\sum_{t=1}^{n}X_{t-1}u_{t}&=n^{-1}\sum_{t=1}^{n}\left(X_{t}^{2}-X_{t-1}^{2}-u_{t}^{2}\right)/2\\&=\left(n^{-1}X_{n}^{2}-n^{-1}\sum_{t=1}^{n}u_{t}^{2}\right)/2\\&=\left(W_{X,n}^{2}\left(1\right)-n^{-1}\sum_{t=1}^{n}u_{t}^{2}\right)/2\\&\rightarrow_{d}\left(B_{X}^{2}\left(1\right)-\sigma_{u}^{2}\right)/2\\&=\left(\omega_{u}^{2}W_{X}\left(1\right)-\sigma_{u}^{2}\right)/2,\end{aligned}$$

where $\sigma_{u}^{2}=Eu_{t}^{2}$. Therefore,

$$n^{-2}\sum_{t=1}^{n}X_{t-1}u_{t}=o_{p}\left(1\right),$$

and

$$n^{-2}\sum_{t=1}^{n}X_{t}^{2}\rightarrow_{d}\int_{0}^{1}B_{X}^{2}\left(r\right)dr. \tag{1}$$

Next, consider $\sum_{t=1}^{n}X_{t}Y_{t}$. First,

$$\begin{aligned}n^{-2}\sum_{t=1}^{n}X_{t-1}Y_{t-1}&=n^{-1}\sum_{t=1}^{n}\left(n^{-1/2}X_{t-1}\right)\left(n^{-1/2}Y_{t-1}\right)\\&=\int_{0}^{1}W_{X,n}\left(r\right)W_{Y,n}\left(r\right)dr\\&\rightarrow_{d}\int_{0}^{1}B_{X}\left(r\right)B_{Y}\left(r\right)dr.\end{aligned}$$

Next,

$$\begin{aligned}n^{-2}\sum_{t=1}^{n}X_{t}Y_{t}&=n^{-2}\sum_{t=1}^{n}\left(X_{t-1}+u_{t}\right)\left(Y_{t-1}+v_{t}\right)\\&=n^{-2}\sum_{t=1}^{n}X_{t-1}Y_{t-1}+n^{-2}\sum_{t=1}^{n}X_{t-1}v_{t}+n^{-2}\sum_{t=1}^{n}Y_{t-1}u_{t}+n^{-2}\sum_{t=1}^{n}u_{t}v_{t}.\end{aligned}$$

Now, $n^{-1}\sum_{t=1}^{n}u_{t}v_{t}\rightarrow_{p}\sigma_{uv}$, where $\sigma_{uv}=Eu_{t}v_{t}$, and, therefore,

$$n^{-2}\sum_{t=1}^{n}u_{t}v_{t}=o_{p}\left(1\right).$$

Further,

$$\begin{aligned} n^{-2}\left|\sum_{t=1}^{n} Y_{t-1}u_t\right| &\leq n^{-2}\sqrt{\sum_{t=1}^{n} Y_{t-1}^2 \sum_{t=1}^{n} u_t^2} \\ &= n^{-1/2}\sqrt{n^{-2}\sum_{t=1}^{n} Y_{t-1}^2 n^{-1}\sum_{t=1}^{n} u_t^2} \\ &= o_p(1). \end{aligned}$$

Similarly,

$$n^{-2}\sum_{t=1}^{n} X_{t-1}v_t = o_p(1).$$

Thus,

$$\begin{aligned} n^{-2}\sum_{t=1}^{n} X_tY_t &= n^{-2}\sum_{t=1}^{n} X_{t-1}Y_{t-1} + o_p(1) \\ &\to_d \int_0^1 B_X(r)\,B_Y(r)\,dr. \end{aligned} \tag{2}$$

The convergence in distribution results in (1) and (2) are joint, and it follows that

$$\begin{aligned} \widehat{\beta}_n &\to_d \frac{\int_0^1 B_X(r)\,B_Y(r)\,dr}{\int_0^1 B_X^2(r)\,dr} \\ &= \xi. \end{aligned}$$

The result holds even if $\{u_t\}$ and $\{v_t\}$ are independent. One could expect that $\widehat{\beta}_n$ would converge in probability to zero, however, it converges in distribution to a random variable ξ and, therefore, is inconsistent. The random variable ξ can be interpreted as a regression coefficient from the "population" or "continuous time" regression of the Brownian motion B_Y against B_X.

Next, consider the usual t-statistic for $H_0 : \beta = 0$:

$$t_{\widehat{\beta}_n} = \widehat{\beta}_n / \left(\frac{s_n^2}{\sum_{t=1}^{n} X_t^2}\right)^{1/2},$$

where s_n^2 is the sample variance of the fitted residuals.

$$\begin{aligned} s_n^2 &= (n-1)^{-1}\sum_{t=1}^{n}\left(Y_t - \widehat{\beta}_n X_t\right)^2 \\ &= (n-1)^{-1}\left(\sum_{t=1}^{n} Y_t^2 + \widehat{\beta}_n^2\sum_{t=1}^{n} X_t^2 - 2\widehat{\beta}_n\sum_{t=1}^{n} X_tY_t\right) \\ &= \frac{n}{n-1}n^{-1}\left(\sum_{t=1}^{n} Y_t^2 - \widehat{\beta}_n\sum_{t=1}^{n} X_tY_t\right), \end{aligned}$$

and, therefore,

$$\begin{aligned} n^{-1}s_n^2 &\to_d \int_0^1 B_Y^2(r)\,dr - \xi\int_0^1 B_X(r)\,B_Y(r)\,dr \\ &= \int_0^1 \left(B_Y(r) - \xi B_X(r)\right)^2 dr. \end{aligned}$$

Lastly,

$$n^{-1/2} t_{\widehat{\beta}_n} = \widehat{\beta}_n / \left(\frac{n^{-1} s_n^2}{n^{-2} \sum_{t=1}^n X_t^2} \right)^{1/2}$$
$$\to_d \xi / \left(\frac{\int_0^1 \left(B_Y(r) - \xi B_X(r)\right)^2 dr}{\int_0^1 B_X^2(r)\, dr} \right)^{1/2}.$$

We conclude that

$$t_{\widehat{\beta}_n} = O_p\left(n^{1/2}\right).$$

Hence, as $n \to \infty$, for any $K > 0$

$$P\left(\left|t_{\widehat{\beta}_n}\right| > K\right) \to 1,$$

and the econometrician will reject $H_0 : \beta = 0$ with the probability approaching 1. This is despite the fact that the two variables X and Y can be independent.

Testing for unit root

Suppose that the scalar process $\{X_t\}$ is generated satisfies the following assumptions:

- $X_t = \rho X_{t-1} + u_t$ $X_0 = 0$.
- $u_t = C(L)\varepsilon_t$.
- $\{\varepsilon_t\}$ is iid, $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2$.
- $\sum_{j=1}^{\infty} j^{1/2} c_j < \infty$, $C(1) \neq 0$.

We are interested in testing

$$H_0 : \rho = 1.$$

against

$$H_0 : |\rho| < 1.$$

Under the null, $X_t = I(1)$, while under the alternative, it is a stationary short memory process.

Consider the regression of X_t against X_{t-1}:

$$\begin{aligned} \widehat{\rho}_n &= \frac{\sum_{t=1}^n X_{t-1} X_t}{\sum_{t=1}^n X_{t-1}^2} \\ &= \rho + \frac{\sum_{t=1}^n X_{t-1} u_t}{\sum_{t=1}^n X_{t-1}^2}. \end{aligned}$$

From the previous section, we know that

$$n^{-2} \sum_{t=1}^n X_{t-1}^2 \to_d \int_0^1 B_X^2(r)\, dr$$
$$= \omega_u^2 \int_0^1 W_X^2(r)\, dr,$$

where W_X is a standard Brownian motion, and

$$n^{-1} \sum_{t=1}^n X_{t-1} u_t \to_d \frac{1}{2}\left(B_X^2(1) - \sigma_u^2\right)$$
$$= \frac{1}{2}\left(\omega_u^2 W_X^2(1) - \sigma_u^2\right)$$
$$= \frac{\omega_u^2}{2}\left(W_X^2(1) - 1\right) + \lambda_u,$$

where

$$\begin{aligned}\lambda_u &= \frac{\omega_u^2 - \sigma_u^2}{2} \\ &= \frac{1}{2}\sum_{h=-\infty}^{\infty}\left(\gamma_u(h) - \sigma_u^2\right) \\ &= \sum_{h=1}^{\infty}\gamma_u(h).\end{aligned}$$

Now, under $H_0 : \rho = 1$,

$$\begin{aligned}n\left(\widehat{\rho}_n - 1\right) &= \frac{n^{-1}\sum_{t=1}^{n} X_{t-1}u_t}{n^{-2}\sum_{t=1}^{n} X_{t-1}^2} \\ &\to_d \frac{\omega_u^2\left(W_X^2(1) - 1\right)/2 + \lambda_u}{\omega_u^2\int_0^1 W_X^2(r)\,dr} \\ &= \frac{\left(W_X^2(1) - 1\right)/2 + \lambda_u/\omega_u^2}{\int_0^1 W_X^2(r)\,dr}.\end{aligned}$$

In the unit root case, the asymptotic distribution depends on functionals of a standard Brownian motion and the nuisance parameters, λ_u and ω_u^2. The convergence rate of $\widehat{\rho}_n$ is faster than the usual $\sqrt{n}$. Next, consider the t statistic for $H_0 : \rho = 1$.

$$T = \left(\widehat{\rho}_n - 1\right)/\left(\frac{\widehat{\sigma}_u^2}{\sum_{t=1}^{n} X_{t-1}^2}\right)^2,$$

where

$$\begin{aligned}\widehat{\sigma}_u^2 &= n^{-1}\sum_{t=1}^{n}\widehat{u_t}^2 \\ &= \sum_{t=1}^{n}\left(X_t - \widehat{\rho}_n X_{t-1}\right)^2 \\ &= n^{-1}\sum_{t=1}^{n}\left(X_t - X_{t-1} - \left(\widehat{\rho}_n - 1\right)X_{t-1}\right)^2 \\ &= n^{-1}\sum_{t=1}^{n} u_t^2 + n\left(\widehat{\rho}_n - 1\right)^2 n^{-2}\sum_{t=1}^{n} X_{t-1}^2 - 2\left(\widehat{\rho}_n - 1\right)n^{-1}\sum_{t=1}^{n} X_{t-1}u_t \\ &= n^{-1}\sum_{t=1}^{n} u_t^2 + O_p\left(n^{-1}\right) \\ &\to_p \sigma_u^2.\end{aligned}$$

Thus,

$$\begin{aligned}T &= n\left(\widehat{\rho}_n - 1\right)/\left(\frac{\widehat{\sigma}_u^2}{n^{-2}\sum_{t=1}^{n} X_{t-1}^2}\right)^{1/2} \\ &\to_d \left(\frac{\left(W_X^2(1) - 1\right)/2 + \lambda_u/\omega_u^2}{\int_0^1 W_X^2(r)\,dr}\right)/\left(\frac{\sigma_u^2}{\omega_u^2\int_0^1 W_X^2(r)\,dr}\right)^{1/2} \\ &= \frac{\omega_u}{\sigma_u}\frac{\left(W_X^2(1) - 1\right)/2 + \lambda_u/\omega_u^2}{\left(\int_0^1 W_X^2(r)\,dr\right)^{1/2}}.\end{aligned}$$

Again, the asymptotic distribution of the statistic depends on the unknown nuisance parameters λ_u, σ_u and ω_u. Phillips (1987) and Phillips and Perron (1988) suggested an adjustment, which leads to an asymptotic distribution free of nuisance parameters. Let $\widehat{\sigma}_u^2$ and $\widehat{\lambda}$ be consistent estimators of σ_u^2 and λ, where λ can be estimated using the Newey-West type estimator:

$$\widehat{\lambda} = \sum_{h=1}^{m_n} \left(1 - \frac{h}{m_n + 1}\right) n^{-1} \sum_{t=h+1}^{n} \widehat{u}_t \widehat{u}_{t-h}.$$

Notice that a consistent estimator of the long-run variance ω_u^2 is

$$\widehat{\omega}_u^2 = \widehat{\sigma}_u^2 + 2\widehat{\lambda}.$$

Consider the following modification of the t statistic.

$$Z_T = \frac{\widehat{\sigma}_u}{\widehat{\omega}_u} T - \frac{\widehat{\lambda}}{\widehat{\omega}_u \left(n^{-2} \sum_{t=1}^{n} X_{t-1}^2\right)^{1/2}}.$$

Under $H_0 : \rho = 1$,

$$Z_T \to_d \frac{1}{2} \frac{W_X^2(1) - 1}{\left(\int_0^1 W_X^2(r)\, dr\right)^{1/2}}.$$

Under the alternative, $|\rho| < 1$, and $\widehat{\rho}_n - 1$ converges in probability to a negative constant. Consequently, under the stationary alternatives, T and Z_T diverge to $-\infty$. One should reject the null of unit root when

$$Z_T < c_\alpha,$$

where c_α is such that

$$P(Z_T < c_\alpha) \overset{H_0:\rho=1}{\to} \alpha$$

Under the null, the distribution is non-standard, however, it is parameter free, and the critical values can be simulated as follows. First, one generates n independent $N(0,1)$ random variables $u_{1,r}^*, \ldots, u_{n,r}^*$ and computes

$$Z_{T,r}^* = \frac{1}{2} \frac{\left(n^{-1/2} \sum_{t=1}^{n} u_{t,r}^*\right)^2 - 1}{\left(n^{-2} \sum_{t=1}^{n} \left(\sum_{s=1}^{t} u_{s,r}^*\right)^2\right)^{1/2}}.$$

One repeats this for $r = 1, \ldots, R$, where R is large. The simulated critical value $c_{\alpha,R}$ is the α quantile of $\left\{Z_{T,1}^*, \ldots, Z_{T,R}^*\right\}$.

While the distribution of Z_T is free of nuisance parameters, it depends on the model. For example, in general one would like to allow for an intercept, $X_t = \mu + X_{t-1} + u_t$. In this case, $\widehat{\rho}_n$ depends on the demeaned X_t:

$$\begin{aligned}
\widehat{\rho}_n &= \frac{\sum_{t=1}^{n} \left(X_{t-1} - \overline{X}_n\right) X_t}{\sum_{t=1}^{n} \left(X_{t-1} - \overline{X}_n\right)^2}, \text{ where} \\
\overline{X}_n &= n^{-1} \sum_{t=1}^{n} X_{t-1}.
\end{aligned}$$

Notice that

$$\begin{aligned} n^{-2}\sum_{t=1}^{n}\left(X_{t-1}-\overline{X}_n\right)^2 &= \int_0^1\left(W_{X,n}(r)-\int_0^1 W_{X,n}(r)\,dr\right)^2 dr \\ &\to_d \int_0^1\left(B_X(r)-\int_0^1 B_X(r)\,dr\right)^2 dr \\ &= \omega_u^2\int_0^1\left(W_X(r)-\int_0^1 W_X(r)\,dr\right)^2 dr \\ &= \omega_u^2\int_0^1 \widetilde{W}_X^2(r)\,dr, \end{aligned}$$

where $\widetilde{W}_X$ is a demeaned standard Brownian motion:

$$\widetilde{W}_X(r) = W_X(r) - \int_0^1 W_X(r)\,dr.$$

Hence, in this case,

$$\begin{aligned} T &= (\widehat{\rho}_n - 1)\Big/\left(\frac{\widehat{\sigma}_u^2}{\sum_{t=1}^n\left(X_{t-1}-\overline{X}_n\right)^2}\right)^{1/2}, \\ Z_T &= \frac{\widehat{\sigma}_u}{\widehat{\omega}_u}T - \frac{\widehat{\lambda}}{\widehat{\omega}_u\left(\sum_{t=1}^n\left(X_{t-1}-\overline{X}_n\right)^2\right)^{1/2}}, \end{aligned}$$

and the asymptotic distribution of Z_T under the null of unit root is given by

$$\frac{1}{2}\frac{\widetilde{W}_X^2(1)-1}{\left(\int_0^1\widetilde{W}_X^2(r)\,dr\right)^{1/2}}.$$