LECTURE 11
LINEAR PROCESSES III: ASYMPTOTIC RESULTS

(Phillips and Solo (1992) and Phillips' Lecture Notes on Stationary and Nonstationary Time Series)

In this lecture, we discuss the LLN and CLT for a linear process $\{X_t\}$ generated as

$$\begin{aligned} X_t &= \sum_{j=0}^{\infty} c_j \varepsilon_{t-j} \\ &= C(L)\varepsilon_t, \end{aligned} \tag{1}$$

where

$$C(L) = \sum_{j=0}^{\infty} c_j L^j,$$

and $\{\varepsilon_t\}$ is a sequence of iid random variables with zero mean and finite variance. The LLN and CLT for $\{X_t\}$ rely only on the LLN and CLT for iid sequences and a certain decomposition of the lag polynomial $C(L)$.

The method also works for more general sequences with ε_t's being independent but not identically distributed (inid) and martingale difference sequences (mds).

Definition 1 *Let $\{\mathcal{F}_t\}$ be an increasing sequence of σ-fields. Then $\{(u_t, \mathcal{F}_t)\}$ is a martingale if $E(u_t|\mathcal{F}_{t-1}) = u_{t-1}$ with probability one. The sequence $\{(\varepsilon_t, \mathcal{F}_t)\}$ is said to be a martingale difference sequence if $E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$ with probability one.*

For $\{u_t\}$, the σ-field $\mathcal{F}_t$ is often taken to be $\sigma(u_t, u_{t-1}, \ldots)$. Suppose that $\{(u_t, \mathcal{F}_t)\}$ is a martingale. Then $\{(u_t - u_{t-1}, \mathcal{F}_t)\}$ is an mds, since $u_t - u_{t-1} = u_t - E(u_t|\mathcal{F}_{t-1})$, and, therefore, $E(u_t - u_{t-1}|\mathcal{F}_{t-1}) = 0$. Note either of the requirements for $\{\varepsilon_t\}$, iid, inid or mds, is stronger than just a WN.

Lemma 1 *(SLLN for inid sequences, White (2001), Corollary 3.9) Let $\{\varepsilon_t\}$ be a sequence of independent random variables such that $\sup_t E|\varepsilon_t|^{1+\delta} < \infty$ for some $\delta > 0$. Then, $n^{-1}\sum_{t=1}^n \varepsilon_t - n^{-1}\sum_{t=1}^n E\varepsilon_t \to_{a.s.} 0$.*

Lemma 2 *(SLLN for mds, White (2001), Exercise 3.77) Let $\{(\varepsilon_t, \mathcal{F}_t)\}$ be an mds such that $\sup_t E|\varepsilon_t|^{2+\delta} < \infty$ for some $\delta > 0$. Then, $n^{-1}\sum_{t=1}^n \varepsilon_t \to_{a.s.} 0$.*

Lemma 3 *(CLT for inid sequences, White (2001), Theorem 5.10) Let $\{\varepsilon_t\}$ be a sequence of independent random variables such that $E\varepsilon_t = 0$ for all t, $\sup_t E|\varepsilon_t|^{2+\delta} < \infty$ for some $\delta > 0$, and for all n sufficiently large $n^{-1}\sum_{t=1}^n E\varepsilon_t^2 > \delta' > 0$. Then, $n^{-1/2}\sum_{t=1}^n \varepsilon_t / \left(n^{-1}\sum_{t=1}^n E\varepsilon_t^2\right)^{1/2} \to_d N(0,1)$.*

Lemma 4 *(CLT for mds, White (2001), Corollary 5.26) Let $\{(\varepsilon_t, \mathcal{F}_t)\}$ be a mds such that $\sup_t E|\varepsilon_t|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose that for all n sufficiently large $n^{-1}\sum_{t=1}^n E\varepsilon_t^2 > \delta' > 0$, and $n^{-1}\sum_{t=1}^n \varepsilon_t^2 - n^{-1}\sum_{t=1}^n E\varepsilon_t^2 \to_p 0$. Then, $n^{-1/2}\sum_{t=1}^n \varepsilon_t / \left(n^{-1}\sum_{t=1}^n E\varepsilon_t^2\right)^{1/2} \to_d N(0,1)$.*

Lemma 5 *(CLT for strictly stationary and ergodic mds) Let $\{(\varepsilon_t, \mathcal{F}_t)\}$ be a strictly stationary and ergodic mds such that $E\varepsilon_t^2 < \infty$. Then $n^{-1/2}\sum_{t=1}^n \varepsilon_t \to_d N\left(0, E\varepsilon_t^2\right)$.*

Beveridge and Nelson (BN) decomposition

First, we discuss an algebraic decomposition of a lag polynomial into long-run and transitory elements. The decomposition was introduced by Beveridge and Nelson (1981).

Lemma 6 *Let $C(L) = \sum_{j=0}^{\infty} c_j L^j$. Then*

(a) *$C(L) = C(1) - (1-L)\widetilde{C}(L)$, where $\widetilde{C}(L) = \sum_{j=0}^{\infty} \widetilde{c}_j L^j$ with $\widetilde{c}_j = \sum_{h=j+1}^{\infty} c_h$.*

(b) *If* $\sum_{j=1}^{\infty} j^{1/2} |c_j| < \infty$*, then* $\sum_{j=0}^{\infty} \widetilde{c}_j^2 < \infty$.

(c) *If* $\sum_{j=1}^{\infty} j |c_j| < \infty$*, then* $\sum_{j=0}^{\infty} |\widetilde{c}_j| < \infty$.

Proof. For part (a), write

$$
\begin{aligned}
\sum_{j=0}^{\infty} c_j L^j \quad = \quad & \sum_{j=0}^{\infty} c_j - \sum_{j=1}^{\infty} c_j \\
& + \left(\sum_{j=1}^{\infty} c_j - \sum_{j=2}^{\infty} c_j \right) L \\
& + \left(\sum_{j=2}^{\infty} c_j - \sum_{j=3}^{\infty} c_j \right) L^2 \\
& + \ldots \\
& + \left(\sum_{j=h}^{\infty} c_j - \sum_{j=h+1}^{\infty} c_j \right) L^h \\
& + \ldots .
\end{aligned}
$$

Rearranging the terms,

$$
\begin{aligned}
\sum_{j=0}^{\infty} c_j L^j \quad = \quad & \sum_{j=0}^{\infty} c_j \\
& - (1 - L) \sum_{j=1}^{\infty} c_j \\
& - (1 - L) \sum_{j=2}^{\infty} c_j L \\
& - \ldots \\
& - (1 - L) \sum_{j=h+1}^{\infty} c_j L^h \\
& - \ldots \\
= \quad & \sum_{j=0}^{\infty} c_j - (1 - L) \sum_{j=0}^{\infty} \left(\sum_{h=j+1}^{\infty} c_h \right) L^j \\
= \quad & C(1) - (1 - L) \widetilde{C}(L) .
\end{aligned}
$$

For part (b),

$$\begin{aligned}
\sum_{j=0}^{\infty} \widetilde{c}_j^2 &= \sum_{j=0}^{\infty} \left(\sum_{h=j+1}^{\infty} c_h \right)^2 \\
&\leq \sum_{j=0}^{\infty} \left(\sum_{h=j+1}^{\infty} |c_h| \right)^2 \\
&= \sum_{j=0}^{\infty} \left(\sum_{h=j+1}^{\infty} |c_h|^{1/2} h^{1/4} |c_h|^{1/2} h^{-1/4} \right)^2 \\
&\leq \sum_{j=0}^{\infty} \left(\sum_{h=j+1}^{\infty} |c_h| h^{1/2} \right) \left(\sum_{h=j+1}^{\infty} |c_h| h^{-1/2} \right) \\
&\leq \left(\sum_{h=0}^{\infty} |c_h| h^{1/2} \right) \sum_{j=0}^{\infty} \sum_{h=j+1}^{\infty} |c_h| h^{-1/2}.
\end{aligned}$$

Next, consider $\sum_{j=0}^{\infty} \sum_{h=j+1}^{\infty} |c_h| h^{-1/2}$. The term $|c_1|$ appears in the sum only once, when $j = 0$. The term $|c_2|$ appears in the sum twice, when $j = 0, 1$. Hence, $|c_h|$ appears when $j = 0, 1, \ldots, h-1$, total h times. Therefore,

$$\begin{aligned}
\sum_{j=0}^{\infty} \sum_{h=j+1}^{\infty} |c_h| h^{-1/2} &= \sum_{j=0}^{\infty} |c_j| j^{-1/2} j \\
&= \sum_{j=0}^{\infty} |c_j| j^{1/2},
\end{aligned}$$

and

$$\sum_{j=0}^{\infty} \widetilde{c}_j^2 \leq \left(\sum_{j=0}^{\infty} |c_j| j^{1/2} \right)^2.$$

For part (c),

$$\begin{aligned}
\sum_{j=0}^{\infty} |\widetilde{c}_j| &= \sum_{j=0}^{\infty} \left| \sum_{h=j+1}^{\infty} c_h \right| \\
&\leq \sum_{j=0}^{\infty} \sum_{h=j+1}^{\infty} |c_h| \\
&= \sum_{j=0}^{\infty} |c_j| j.
\end{aligned}$$

■

Notice that the assumptions $\sum_{j=1}^{\infty} j^{1/2} |c_j| < \infty$ and $\sum_{j=1}^{\infty} j |c_j| < \infty$ are stronger than finiteness of the long-run variance $\sum_{j=1}^{\infty} |c_j| < \infty$.

According to the BN decomposition, if $\{X_t\}$ is a linear process, then

$$\begin{aligned}
X_t &= C(L) \varepsilon_t \\
&= C(1) \varepsilon_t - (1 - L) \widetilde{C}(L) \varepsilon_t \\
&= C(1) \varepsilon_t - (\widetilde{\varepsilon}_t - \widetilde{\varepsilon}_{t-1}), \qquad (2)
\end{aligned}$$

where

$$\widetilde{\varepsilon}_t = \widetilde{C}(L)\,\varepsilon_t.$$

Furthermore, $\widetilde{\varepsilon}_t$ has finite variance provided that $\sum_{j=0}^{\infty} j^{1/2}|c_j| < \infty$. The first summand on the right-hand side of (2), $C(1)\,\varepsilon_t$, is the long-run component, and $\widetilde{\varepsilon}_t - \widetilde{\varepsilon}_{t-1}$ is transient.

The similar decomposition exists in the vector case. The transient component has finite variance if $\sum_{j=1}^{\infty} j^{1/2}\|C_j\| < \infty$:

$$\begin{aligned}
\sum_{j=0}^{\infty}\left\|\widetilde{C}_j\right\|^2 &= \sum_{j=0}^{\infty}\left\|\sum_{h=j+1}^{\infty} C_h\right\|^2 \\
&\leq \sum_{j=0}^{\infty}\left(\sum_{h=j+1}^{\infty}\|C_j\|\right)^2 \\
&= \sum_{j=0}^{\infty}\left(\sum_{h=j+1}^{\infty}\|C_h\|^{1/2} h^{1/4}\|C_h\|^{1/2} h^{-1/4}\right)^2 \\
&\leq \left(\sum_{j=0}^{\infty} j^{1/2}\|C_j\|\right)^2.
\end{aligned}$$

The condition in part (c) of Theorem 6, becomes $\sum_{j=1}^{\infty} j\|C_j\| < \infty$ in the vector case.

WLLN

Suppose that

- $X_t = C(L)\,\varepsilon_t$.
- $\{\varepsilon_t\}$ is a sequence of iid random variables with $E|\varepsilon_t| < \infty$ and $E\varepsilon_t = 0$.
- $C(L)$ satisfies $\sum_{j=1}^{\infty} j|c_j| < \infty$.

We will show that under these conditions

$$n^{-1}\sum_{t=1}^{n} X_t \to_p 0.$$

The key is the BN decomposition (2). Notice that $\sum_{t=1}^{n}(\widetilde{\varepsilon}_t - \widetilde{\varepsilon}_{t-1})$ is a so called telescoping sum:

$$\begin{aligned}
\sum_{t=1}^{n}(\widetilde{\varepsilon}_t - \widetilde{\varepsilon}_{t-1}) &= (\tilde{\varepsilon}_1 - \tilde{\varepsilon}_0) + (\tilde{\varepsilon}_2 - \tilde{\varepsilon}_1) + (\tilde{\varepsilon}_3 - \tilde{\varepsilon}_2) + \ldots + (\tilde{\varepsilon}_n - \tilde{\varepsilon}_{n-1}) \\
&= \widetilde{\varepsilon}_n - \widetilde{\varepsilon}_0,
\end{aligned}$$

so that

$$n^{-1}\sum_{t=1}^{n} X_t = C(1)\,n^{-1}\sum_{t=1}^{n}\varepsilon_t - n^{-1}(\widetilde{\varepsilon}_n - \widetilde{\varepsilon}_0).$$

Due to $\sum_{j=1}^{\infty} j|c_j| < \infty$, $|C(1)| < \infty$. Hence, by the WLLN for iid sequences,

$$C(1)\,n^{-1}\sum_{t=1}^{n}\varepsilon_t \to_p 0.$$

Next,

$$P\left(n^{-1}\left|\widetilde{\varepsilon}_t\right|>\delta\right)\leq\frac{E\left|\widetilde{\varepsilon}_t\right|}{n\delta},$$

and

$$\begin{aligned}
E\left|\widetilde{\varepsilon}_t\right| &= E\left|\sum_{j=0}^{\infty}\widetilde{c}_j\varepsilon_{t-j}\right| \\
&\leq \sum_{j=0}^{\infty}\left|\widetilde{c}_j\right|E\left|\varepsilon_{t-j}\right| \\
&= E\left|\varepsilon_0\right|\sum_{j=0}^{\infty}\left|\widetilde{c}_j\right| \\
&< \infty,
\end{aligned}$$

provided that $\sum_{j=1}^{\infty} j\left|c_j\right|<\infty$. Therefore,

$$n^{-1}\left(\widetilde{\varepsilon}_n-\widetilde{\varepsilon}_0\right)\to_p 0.$$

If we assume that $E\varepsilon_t^2<\infty$, then we can replace $\sum_{j=1}^{\infty} j\left|c_j\right|<\infty$ with $\sum_{j=1}^{\infty} j^{1/2}\left|c_j\right|<\infty$, since

$$P\left(n^{-1}\left|\widetilde{\varepsilon}_t\right|>\delta\right)\leq\frac{E\widetilde{\varepsilon}_t^2}{n^2\delta^2},$$

and $E\widetilde{\varepsilon}_t^2<\infty$ provided that $\sum_{j=1}^{\infty} j^{1/2}\left|c_j\right|<\infty$ holds.

We can prove similar WLLNs with $\{\varepsilon_t\}$ being inid or an mds by using the corresponding LLNs for inid or mds. For example, the result holds if $\{\varepsilon_t\}$ is inid, $\sup_t E\left|\varepsilon_t\right|^{1+\delta}<\infty$ for some $\delta>0$, and $\sum_{j=1}^{\infty} j\left|c_j\right|<\infty$. If $\{(\varepsilon_t,\mathcal{F}_t)\}$ is an mds, the result holds with $\sup_t E\left|\varepsilon_t\right|^{2+\delta}<\infty$ for some $\delta>0$, and $\sum_{j=1}^{\infty} j^{1/2}\left|c_j\right|<\infty$.

So far we assumed that $EX_t=0$. One can modify the first assumption so that $X_t=\mu+C(L)\varepsilon_t$, where μ is the mean of X_t. In this case, under the same set of conditions, we have $n^{-1}\sum_{t=1}^{n} X_t\to_p\mu$. For example, $AR(1)$ process with mean μ is given by $(1-\phi L)(X_t-\mu)=\varepsilon_t$. If $|\phi|<1$, then the sample average of X_t converges in probability to μ provided that the corresponding moment restrictions hold.

CLT

Suppose that

- $X_t=C(L)\varepsilon_t$.
- $\{\varepsilon_t\}$ is a sequence of iid random variables with $E\left|\varepsilon_t\right|^2=\sigma^2<\infty$ and $E\varepsilon_t=0$.
- $C(L)$ satisfies $\sum_{j=1}^{\infty} j^{1/2}\left|c_j\right|<\infty$.
- $C(1)\neq 0$.

The BN decomposition allows us to write

$$\begin{aligned}
n^{-1/2}\sum_{t=1}^{n}X_t &= C(1)\,n^{-1/2}\sum_{t=1}^{n}\varepsilon_t-n^{-1/2}\left(\widetilde{\varepsilon}_n-\widetilde{\varepsilon}_0\right)\\
&= C(1)\,n^{-1/2}\sum_{t=1}^{n}\varepsilon_t+o_p(1)\\
&\to_d C(1)\,N\left(0,\sigma^2\right)\\
&= N\left(0,\sigma^2C(1)^2\right).
\end{aligned}$$

Here, convergence in distribution is by the CLT for iid random variables. The approach illustrates why in the serially correlated case the asymptotic variance depends on the long-run variance of $\{X_t\}$. Again, the approach can be extended to the case where $\{\varepsilon_t\}$ is inid of mds.

In the vector case, suppose that $\{\varepsilon_t\}$ is a sequence of iid k-vectors with $E\varepsilon_t = 0$, and $E\varepsilon_t\varepsilon_t' = \Sigma$, a finite matrix. Let $X_t = C(L)\varepsilon_t$, and $\sum_{j=0}^{\infty} j^{1/2}\|C_j\| < \infty$, $C(1) \neq 0$. Since $n^{-1/2}\sum_{t=1}^{n}\varepsilon_t \to_d N(0,\Sigma)$, we have that

$$n^{-1/2}\sum_{t=1}^{n} X_t \to_d N\left(0, C(1)\Sigma C(1)'\right).$$

Convergence of sample variances

Estimators of coefficients in the linear regression model involve second sample moments $n^{-1}\sum_{t=1}^{n} X_t X_t'$. Here, we discuss convergence of sample second moments when $\{X_t\}$ is a linear process. We assume that $\{X_t\}$ is a scalar linear process satisfying the same assumptions as in the previous section. Write

$$\begin{aligned}
X_t^2 &= (C(L)\varepsilon_t)^2 \\
&= \left(\sum_{j=0}^{\infty} c_j\varepsilon_{t-j}\right)^2 \\
&= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_j c_l \varepsilon_{t-j}\varepsilon_{t-l} \\
&= \sum_{j=0}^{\infty} c_j^2\varepsilon_{t-j}^2 + 2\sum_{j=0}^{\infty}\sum_{l>j} c_j c_l \varepsilon_{t-j}\varepsilon_{t-l} \\
&= \sum_{j=0}^{\infty} c_j^2\varepsilon_{t-j}^2 + 2\sum_{h=1}^{\infty}\sum_{j=0}^{\infty} c_j c_{j+h}\varepsilon_{t-j}\varepsilon_{t-j-h} \text{ (change of variable } l = j+h\text{, so that } h = 1,2,\dots) \\
&= B_0(L)\varepsilon_t^2 + 2\sum_{h=1}^{\infty} B_h(L)\varepsilon_t\varepsilon_{t-h},
\end{aligned}$$

where for $h = 0, 1, \dots,$

$$\begin{aligned}
B_h(L) &= \sum_{j=0}^{\infty} b_{h,j}L^j \\
&= \sum_{j=0}^{\infty} c_j c_{j+h} L^j.
\end{aligned}$$

Thus,

$$\begin{aligned}
B_0(L) &= \sum_{j=0}^{\infty} b_{0,j}L^j = \sum_{j=0}^{\infty} c_j^2 L^j. \\
B_1(L) &= \sum_{j=0}^{\infty} b_{1,j}L^j = \sum_{j=0}^{\infty} c_j c_{j+1} L^j. \\
&\dots
\end{aligned}$$

The BN decomposition of $B_h(L)$ is

$$B_h(L) = B_h(1) - (1-L)\widetilde{B}_h(L), \tag{3}$$

where

$$\begin{aligned}
\widetilde{B}_h(L) &= \sum_{j=0}^{\infty} \widetilde{b}_{h,j} L^j, \\
\widetilde{b}_{h,j} &= \sum_{l=j+1}^{\infty} b_{h,l} \\
&= \sum_{l=j+1}^{\infty} c_l c_{l+h}.
\end{aligned}$$

The BN decomposition of $B_h(L)$ is valid provided that $\sum_{j=1}^{\infty} j^{1/2} |c_j| < \infty$:

$$\begin{aligned}
\sum_{j=0}^{\infty} \widetilde{b}_{h,j}^2 &= \sum_{j=0}^{\infty} \left(\sum_{l=j+1}^{\infty} c_l c_{l+h} \right)^2 \\
&= \sum_{j=0}^{\infty} \left(\sum_{l=j+1}^{\infty} l^{1/4} c_l c_{l+h} l^{-1/4} \right)^2 \\
&\leq \sum_{j=0}^{\infty} \left(\sum_{l=j+1}^{\infty} l^{1/2} c_l^2 \right) \left(\sum_{l=j+1}^{\infty} c_{l+h}^2 l^{-1/2} \right) \\
&\leq \left(\sum_{l=1}^{\infty} l^{1/2} c_l^2 \right) \left(\sum_{j=0}^{\infty} \sum_{l=j+1}^{\infty} c_{l+h}^2 l^{-1/2} \right) \\
&= \left(\sum_{l=1}^{\infty} l^{1/2} c_l^2 \right) \left(\sum_{l=0}^{\infty} c_{l+h}^2 l^{1/2} \right) \\
&\leq \left(\sum_{l=1}^{\infty} l^{1/2} c_l^2 \right)^2,
\end{aligned}$$

and $\sum_{l=1}^{\infty} l^{1/2} c_l^2$ is finite provided that $\sum_{l=1}^{\infty} l^{1/2} |c_l|$ is finite:

$$\begin{aligned}
\sum_{l=1}^{\infty} l^{1/2} c_l^2 &= \sum_{l=1}^{\infty} l^{1/2} |c_l|^2 \\
&\leq \sup_j |c_j| \sum_{l=1}^{\infty} l^{1/2} |c_l| < \infty,
\end{aligned}$$

where $\sup_j |c_j| < \infty$ because $\sum_{l=1}^{\infty} l^{1/2} |c_l| < \infty$ and therefore $|c_l| \to 0$ as $l \to \infty$.

Thus, we have

$$\begin{aligned}
X_t^2 &= B_0(L)\varepsilon_t^2 + 2\sum_{h=1}^{\infty} B_h(L)\, \varepsilon_t \varepsilon_{t-h} \\
&= B_0(1)\varepsilon_t^2 + 2\sum_{h=1}^{\infty} B_h(1)\, \varepsilon_t \varepsilon_{t-h} \\
&\quad - (1-L)\left(\widetilde{B}_0(L)\, \varepsilon_t^2 + 2\sum_{h=1}^{\infty} \widetilde{B}_h(L)\, \varepsilon_t \varepsilon_{t-h} \right) \\
&= \left(\sum_{j=0}^{\infty} c_j^2 \right) \varepsilon_t^2 + u_t - (1-L)\widetilde{v}_t,
\end{aligned}$$

where

$$
\begin{aligned}
u_t &= \varepsilon_t \left(2 \sum_{h=1}^{\infty} B_h(1) \varepsilon_{t-h} \right), \\
\widetilde{v}_t &= \widetilde{B}_0(L) \varepsilon_t^2 + 2 \sum_{h=1}^{\infty} \widetilde{B}_h(L) \varepsilon_t \varepsilon_{t-h}.
\end{aligned}
$$

We have

$$
n^{-1} \sum_{t=1}^{n} X_t^2 = \left(\sum_{j=0}^{\infty} c_j^2 \right) n^{-1} \sum_{t=1}^{n} \varepsilon_t^2 + n^{-1} \sum_{t=1}^{n} u_t - n^{-1} (\widetilde{v}_n - \widetilde{v}_0).
$$

We will show next that

$$
n^{-1} \sum_{t=1}^{n} u_t \to_{a.s.} 0.
$$

Let $\mathcal{F}_t = \sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$. We have that $\{(u_t, \mathcal{F}_t)\}$ is an mds.

Lemma 7 *(White (2001), Theorem 3.76) Let $\{(u_t, \mathcal{F}_t)\}$ be an mds. If for some $r \geq 1$, $\sum_{t=1}^{\infty} E|u_t|^{2r}/t^{1+r} < \infty$, then $n^{-1}\sum_{t=1}^{n} u_t \to_{a.s.} 0$.*

We will verify that $\{u_t\}$ satisfies the condition of the above lemma. Set $r = 1$. The condition is satisfied if $\sup_t E u_t^2 < \infty$, since $\sum_{t=1}^{\infty} t^{-2} < \infty$.

$$
\begin{aligned}
E u_t^2 &= 4\sigma^2 E \left(\sum_{h=1}^{\infty} B_h(1) \varepsilon_{t-h} \right)^2 \\
&= 4\sigma^4 \left(\sum_{h=1}^{\infty} B_h(1)^2 \right).
\end{aligned}
$$

Next,

$$
\begin{aligned}
\sum_{h=1}^{\infty} B_h(1)^2 &= \sum_{h=1}^{\infty} \left(\sum_{j=0}^{\infty} c_j c_{j+h} \right)^2 \\
&\leq \sum_{h=1}^{\infty} \left(\sum_{j=0}^{\infty} c_j^2 \right) \sum_{j=0}^{\infty} c_{j+h}^2 \\
&= \left(\sum_{j=0}^{\infty} c_j^2 \right) \sum_{h=1}^{\infty} \sum_{j=0}^{\infty} c_{j+h}^2 \\
&= \left(\sum_{j=0}^{\infty} c_j^2 \right) \sum_{j=1}^{\infty} j c_j^2,
\end{aligned}
$$

and, by the same argument as on page 3 of Lecture 10, $\sum_{j=1}^{\infty} j^{1/2} |c_j| < \infty$ implies that $\sum_{j=1}^{\infty} j c_j^2 < \infty$ as well.

As before, one can show that

$$
n^{-1} (\widetilde{v}_n - \widetilde{v}_0) \to_p 0,
$$

provided that $\sum_{j=1}^{\infty} j^{1/2} |c_j| < \infty$.

Lastly, by the WLLN for iid sequences,

$$n^{-1} \sum_{t=1}^{n} \varepsilon_t^2 \to_p \sigma^2.$$

Therefore,

$$\begin{aligned} n^{-1} \sum_{t=1}^{n} X_t^2 &\to_p \sigma^2 \sum_{j=0}^{\infty} c_j^2 \\ &= EX_t^2. \end{aligned}$$