LECTURE 10

LINEAR PROCESSES II: SPECTRAL DENSITY, LAG OPERATOR, ARMA

In this lecture, we continue to discuss covariance stationary processes.

Spectral density

(Gourieroux and Monfort (1990), Ch. 5; Hamilton (1994), Ch. 6)

A convenient way to represent the sequence of autocovariances $\{\gamma(j) : j = 0, 1, \ldots\}$ of a covariance stationary process is by the means of the spectral density or spectrum. The spectral density is defined as follows.

$$f(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma(j) e^{-i\lambda j},$$

where $i = \sqrt{-1}$. Notice that, since $\gamma(j) = \gamma(-j)$, it follows that the spectral density is real valued.

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi} \left(\gamma(0) + \sum_{j=-\infty}^{-1} \gamma(j) e^{-i\lambda j} + \sum_{j=1}^{\infty} \gamma(j) e^{-i\lambda j} \right) \\ &= \frac{1}{2\pi} \left(\gamma(0) + \sum_{j=1}^{\infty} \gamma(-j) e^{i\lambda j} + \sum_{j=1}^{\infty} \gamma(j) e^{-i\lambda j} \right) \\ &= \frac{1}{2\pi} \left(\gamma(0) + \sum_{j=1}^{\infty} \gamma(j) \left(e^{i\lambda j} + e^{-i\lambda j} \right) \right). \end{aligned}$$

Next, $e^{i\lambda j} = \cos(\lambda j) + i \sin(\lambda j)$, $e^{-i\lambda j} = \cos(\lambda j) - i \sin(\lambda j)$, and, therefore,

$$e^{i\lambda j} + e^{-i\lambda j} = 2\cos(\lambda j).$$

Hence,

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi} \left(\gamma(0) + 2 \sum_{j=1}^{\infty} \gamma(j) \cos(\lambda j) \right) \\ &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma(j) \cos(\lambda j). \end{aligned}$$

Since $\cos(\lambda j) = \cos(-\lambda j)$, the spectral density is symmetric around zero. Furthermore, since cos is a periodic function with the period 2π, the range of values of the spectral density is determined by the values of $f(\lambda)$ for $0 \leq \lambda \leq \pi$.

The autocovariance function and spectral density are equivalent, as it follows from the result below.

Theorem 1 *Suppose that* $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. *Then* $\gamma(j) = \int_{-\pi}^{\pi} f(\lambda) e^{i\lambda j} d\lambda$.

Proof.

$$\begin{aligned} \int_{-\pi}^{\pi} f(\lambda) e^{i\lambda j} d\lambda &= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\lambda h} \right) e^{i\lambda j} d\lambda \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \int_{-\pi}^{\pi} e^{i\lambda(j-h)} d\lambda, \end{aligned}$$

where summation and integration can be interchanged because $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. Next,

$$\int_{-\pi}^{\pi} e^{i\lambda(j-h)} d\lambda = 2\pi \text{ if } j = h.$$

For $j \neq h$,

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{i\lambda(j-h)} d\lambda &= \int_{-\pi}^{\pi} \left(\cos\left(\lambda\left(j-h\right)\right) + i \sin\left(\lambda\left(j-h\right)\right)\right) d\lambda \\
&= \left. \frac{\sin\left(\lambda\left(j-h\right)\right)}{j-h} \right|_{-\pi}^{\pi} - i \left. \frac{\cos\left(\lambda\left(j-h\right)\right)}{j-h} \right|_{-\pi}^{\pi} \\
&= \frac{\sin\left(\pi\left(j-h\right)\right) - \sin\left(-\pi\left(j-h\right)\right)}{j-h} \\
&\quad -i \frac{\cos\left(\pi\left(j-h\right)\right) - \cos\left(-\pi\left(j-h\right)\right)}{j-h}.
\end{aligned}$$

However, since cos and sin are periodic with the period 2π,

$$\begin{aligned}
\cos\left(\pi\left(j-h\right)\right) &= \cos\left(-\pi\left(j-h\right) + 2\pi\left(j-h\right)\right) \\
&= \cos\left(-\pi\left(j-h\right)\right).
\end{aligned}$$

Therefore,

$$\int_{-\pi}^{\pi} e^{i\lambda(j-h)} d\lambda = 0 \text{ if } j \neq h.$$

■

The result of Theorem 1 implies in particular that

$$\gamma\left(0\right) = \int_{-\pi}^{\pi} f\left(\lambda\right) d\lambda.$$

Thus, the area under the spectral density function of X_t between $-\pi$ and π gives the variance of X_t.

The argument λ of $f\left(\lambda\right)$ is called the frequency. Notice that if $\{X_t\}$ is covariance stationary with absolutely summable autocovariances, the long-run variance is determined by the spectral density at the zero frequency.

$$\begin{aligned}
\omega_X &= \lim_{n\to\infty} Var\left(n^{-1/2} \sum_{t=1}^{n} X_t\right) \\
&= \sum_{h=-\infty}^{\infty} \gamma\left(h\right) \\
&= 2\pi f\left(0\right).
\end{aligned}$$

Next, we discuss how linear (MA) transformations of a covariance stationary process affect the spectral density and long-run variance.

Theorem 2 *Let $\{X_t\}$ be a covariance stationary process with the autocovariance function γ_X such that $\sum_{j=-\infty}^{\infty} |\gamma_X(j)| < \infty$. Define $Y_t = \sum_{j=0}^{\infty} c_j X_{t-j}$, where $\sum_{j=0}^{\infty} c_j^2 < \infty$. Then $\{Y_t\}$ is covariance stationary and its spectral density is given by $f_Y\left(\lambda\right) = \left|\sum_{j=0}^{\infty} c_j e^{-i\lambda j}\right|^2 f_X\left(\lambda\right)$, where f_X is the spectral density of $\{X_t\}$.*

Proof.

$$
\begin{aligned}
Cov\left(Y_t, Y_{t-h}\right) &= Cov\left(\sum_{j=0}^{\infty} c_j X_{t-j}, \sum_{j=0}^{\infty} c_j X_{t-h-j}\right) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_j c_k Cov\left(X_{t-j}, X_{t-h-k}\right) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_j c_k \gamma_X\left(h+k-j\right).
\end{aligned}
$$

Hence, $Cov\left(Y_t, Y_{t-h}\right)$ is independent of t. Furthermore, by the same argument as on pages 1-2 of Lecture 9,

$$
\begin{aligned}
\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_j c_k \gamma_X\left(h+k-j\right) &\leq 2\left(\sum_{j=0}^{\infty} c_j^2\right)\sum_{h=0}^{\infty}\left|\gamma_X\left(h\right)\right| \\
&< \infty.
\end{aligned}
$$

Therefore, $\{Y_t\}$ is covariance stationary.

Next,

$$
\begin{aligned}
f_Y\left(\lambda\right) &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} Cov\left(Y_t, Y_{t-h}\right)e^{-i\lambda h} \\
&= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_j c_k \gamma_X\left(h+k-j\right)e^{-i\lambda h} \\
&= \sum_{j=0}^{\infty} c_j e^{-i\lambda j}\sum_{k=0}^{\infty} c_k e^{i\lambda k}\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_X\left(h+k-j\right)e^{-i\lambda(h+k-j)} \\
&= \left(\sum_{j=0}^{\infty} c_j e^{-i\lambda j}\right)\left(\sum_{j=0}^{\infty} c_j e^{i\lambda j}\right) f_X\left(\lambda\right) \\
&= \left|\sum_{j=0}^{\infty} c_j e^{-i\lambda j}\right|^2 f_X\left(\lambda\right).
\end{aligned}
$$

The last equality follows because

$$
\begin{aligned}
\sum_j c_j e^{-i\lambda j} &= \sum_j c_j\left(\cos\left(\lambda j\right) - i\sin\left(\lambda j\right)\right) \\
&= \sum_j c_j\cos\left(\lambda j\right) - i\sum_j c_j\sin\left(\lambda j\right).
\end{aligned}
$$

Its complex conjugate is

$$
\begin{aligned}
\sum_j c_j\cos\left(\lambda j\right) + i\sum_j c_j\sin\left(\lambda j\right) &= \sum_j c_j\left(\cos\left(\lambda j\right) + i\sin\left(\lambda j\right)\right) \\
&= \sum_j c_j e^{i\lambda j},
\end{aligned}
$$

and hence

$$\begin{aligned}\left|\sum_{j=0}^{\infty} c_j e^{-i\lambda j}\right|^2 &= \left(\sum_{j=0}^{\infty} c_j \cos(\lambda j)\right)^2 + \left(\sum_{j=0}^{\infty} c_j \sin(\lambda j)\right)^2 \\ &= \left(\sum_{j=0}^{\infty} c_j e^{-i\lambda j}\right)\left(\sum_{j=0}^{\infty} c_j e^{i\lambda j}\right).\end{aligned}$$

■

In the above theorem, the spectral density at the zero frequency and, as a result, the long-run variance is finite if $\sum_{j=0}^{\infty}|c_j| < \infty$. However, absolute summability, $\sum_{j=0}^{\infty}|c_j| < \infty$, is a stronger assumption than square summability, $\sum_{j=0}^{\infty} c_j^2 < \infty$, as we show next. Suppose $\sum_{j=0}^{\infty}|c_j| < \infty$. First, $\sum_{j=0}^{\infty}|c_j| < \infty$ implies that $c_j \to 0$ as $j \to \infty$. Therefore, the sequence $\{c_j\}$ is uniformly bounded. Next, $\sum_{j=0}^{\infty} c_j^2 \leq \sup_j |c_j| \sum_{j=0}^{\infty}|c_j| < \infty$.

Suppose that $\{X_t\}$ is covariance stationary and purely indeterministic. Then it has the MA(∞) representation

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \tag{1}$$

where $\{\varepsilon_t\}$ is a WN, and $\sum_{j=0}^{\infty} a_j^2 < \infty$. Let $Var(\varepsilon_t) = \sigma^2$. Since the spectrum of a WN process is flat:

$$f(\lambda) = \frac{\sigma^2}{2\pi} \text{ for all } \lambda,$$

Theorem 2 implies that

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} a_j e^{-i\lambda j}\right|^2,$$

and the long-run variance of $\{X_t\}$ is

$$\begin{aligned}\omega_X &= 2\pi f_X(0) \\ &= \sigma^2\left(\sum_{j=0}^{\infty} a_j\right)^2.\end{aligned}$$

If we take (1) as the generating mechanism, the condition $\sum_{j=0}^{\infty} a_j^2 < \infty$ ensures that $\{X_t\}$ is covariance stationary. However, the sufficient condition for the long-run variance to be finite is $\sum_{j=0}^{\infty}|a_j| < \infty$. If the last condition fails, we can have that $\sum_{j=-\infty}^{\infty}|\gamma_X(j)| = \infty$. Such a process is called long memory. If $\sum_{j=0}^{\infty} a_j^2 < \infty$ holds for a long memory process, then its autocovariance function converges to zero, however, at the rate that is too slow for the long-run variance to be finite.

Let $\{Y_t\}$ be as defined in Theorem 2. Then its spectral density and long-run variance are given by

$$\begin{aligned}f_Y(\lambda) &= \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} a_j e^{-i\lambda j}\right|^2 \left|\sum_{j=0}^{\infty} c_j e^{-i\lambda j}\right|^2, \\ \omega_Y &= \sigma^2\left(\sum_{j=0}^{\infty} a_j\right)^2\left(\sum_{j=0}^{\infty} c_j\right)^2.\end{aligned}$$

Lag operator

(Gourieroux and Monfort (1990), Ch. 5; Hamilton (1994), Ch. 2)

The lag operator L transforms the process $\{X_t\}$ into itself such that

$$\begin{aligned} LX_t &= X_{t-1}, \\ L^2X_t &= LLX_t = LX_{t-1} = X_{t-2}, \\ & \dots \\ L^hX_t &= X_{t-h}. \end{aligned}$$

The lag polynomial $C(L) = \sum_{j=0}^{\infty} c_j L^j$ transforms $\{X_t\}$ into another process $\{Y_t\}$ such that

$$\begin{aligned} Y_t &= C(L) X_t \\ &= \sum_{j=0}^{\infty} c_j L^j X_t \\ &= \sum_{j=0}^{\infty} c_j X_{t-j}. \end{aligned}$$

Let $A(L) = \sum_{j=0}^{\infty} a_j L^j$ and $B(L) = \sum_{j=0}^{\infty} b_j L^j$. Then

$$A(L) + B(L) = \sum_{j=0}^{\infty} (a_j + b_j) L^j,$$

and

$$\begin{aligned} A(L)B(L) &= \left(\sum_{j=0}^{\infty} a_j L^j \right) \left(\sum_{j=0}^{\infty} b_j L^j \right) \\ &= \sum_{j=0}^{\infty} \sum_{h=0}^{\infty} a_j b_h L^{j+h} \\ &= a_0b_0 + (a_0b_1 + b_0a_1) L + (a_1b_1 + a_0b_2 + a_2b_0) L^2 + \dots. \end{aligned}$$

We have that

$$\begin{aligned} A(L) + B(L) &= B(L) + A(L), \\ A(L)B(L) &= B(L)A(L). \end{aligned}$$

Under certain conditions, a lag polynomial can be inverted. The inverse of a lag polynomial $C(L)$ is another lag polynomial, say $B(L)$ such that

$$C(L) B(L) = 1, \tag{2}$$

so we can write

$$C(L)^{-1} = B(L).$$

Inversion of lag polynomials is important for the following reason. Consider the following *autoregressive process of order 1* (AR(1)):

$$X_t = cX_{t-1} + \varepsilon_t. \tag{3}$$

This process is generated recursively given some exogenous white noise process $\{\varepsilon_t\}$, a starting value X_0 (a random variable with the variance equal to $Var(X_t)$ to be determined later), and the coefficient c:

$$\begin{aligned} X_1 &= cX_0 + \varepsilon_1, \\ X_2 &= cX_1 + \varepsilon_2, \end{aligned}$$

and etc. Thus, the process $\{X_t\}$ is an endogenous solution to the *difference equation* (3). This difference equation can be also written as $X_t - cX_{t-1} = \varepsilon_t$ or

$$\begin{aligned} C(L) X_t &= \varepsilon_t, \\ C(L) &= 1 - cL. \end{aligned}$$

If $C(L)$ can be inverted, then the solution can be written as $X_t = C(L)^{-1} \varepsilon_t$. Thus, it is important to determine under what conditions a polynomial in lag operator can be inverted and how to compute the coefficients of its inverse.

Consider first a polynomial of order 1. Without loss of generality, we can set the coefficient associated with L^0 as $c_0 = 1$:[1]

$$C(L) = 1 - cL.$$

Suppose that $|c| < 1$. Then, we can define the inverse of $1 - cL$ as follows.

$$(1 - cL)^{-1} = 1 + cL + c^2 L^2 + \dots. \tag{4}$$

This definition satisfies (2) since

$$(1 - cL) \sum_{j=0}^{\infty} c^j L^j = 1.$$

The definition (4) is not the only solution that satisfies (2). Adding to it the term Vc^t, where V is some random variable, satisfies it as well, since

$$\begin{aligned} (1 - cL) Vc^t &= Vc^t - VcLc^t \\ &= Vc^t - Vcc^{t-1} \\ &= 0. \end{aligned}$$

However, if we take $(1 - cL)^{-1} = \sum_{j=0}^{\infty} c^j L^j + Vc^t$, then $(1 - cL)^{-1} \varepsilon_t = \sum_{j=0}^{\infty} c^j \varepsilon_{t-j} + Vc^t \varepsilon_t$ and is not a covariance stationary process. Therefore, we restrict $V = 0$.

Next, consider a lag polynomial of order 2:

$$C(L) = 1 - c_1 L - c_2 L^2.$$

We can factor the polynomial as

$$1 - c_1 L - c_2 L^2 = (1 - \lambda_1 L)(1 - \lambda_2 L) \tag{5}$$

where

$$\begin{aligned} c_1 &= \lambda_1 + \lambda_2, \\ c_2 &= -\lambda_1 \lambda_2. \end{aligned}$$

Another way to find λ_1 and λ_2 is as follows. Let z_1 and z_2 be the solutions (possibly complex) to

$$1 - c_1 z - c_2 z^2 = 0. \tag{6}$$

We can write (6) as

$$1 - c_1 z - c_2 z^2 = (z_1 - z)(z_2 - z),$$

and by comparing this with (5), we obtain

$$\lambda_1 = z_1^{-1} \text{ and } \lambda_2 = z_2^{-1}.$$

[1] This is due to the Wold decomposition result. Alternatively, this can be viewed as a normalization: multiplying all coefficients by some constant $c_0 \neq 0$ affects only the variance of the process.

The polynomial in (5) can be inverted provided that $|\lambda_1| < 1$ and $|\lambda_2| < 1$. These conditions are equivalent to the condition that all the roots of the polynomial in (6) are outside the unit circle:

$$|z| > 1.$$

If this condition is satisfied then

$$\begin{aligned}\left(1 - c_1 L - c_2 L^2\right)^{-1} &= (1 - \lambda_1 L)^{-1} (1 - \lambda_2 L)^{-1} \\ &= \left(\sum_{j=0}^{\infty} \lambda_1^j L^j\right) \left(\sum_{j=0}^{\infty} \lambda_2^j L^j\right).\end{aligned}$$

Alternatively, write

$$\frac{1}{1 - \lambda_1 L} \frac{1}{1 - \lambda_2 L} = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L}\right).$$

Then,

$$\begin{aligned}\left(1 - c_1 L - c_2 L^2\right)^{-1} &= \frac{1}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} \left(\lambda_1 \lambda_1^j L^j - \lambda_2 \lambda_2^j L^j\right) \\ &= \frac{1}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} \left(\lambda_1^{j+1} - \lambda_2^{j+1}\right) L^j.\end{aligned}$$

The result can be extended to a lag polynomial of order p,

$$C(L) = 1 - c_1 L - \ldots - c_p L^p,$$

since we can factor it as

$$1 - c_1 L - \ldots - c_p L^p = \prod_{j=1}^{p} (1 - \lambda_j L),$$

where λ_j's satisfy

$$(1 - c_1 z - \ldots - c_p z^p) = \prod_{j=1}^{p} (1 - \lambda_j z).$$

The polynomial can be inverted provided that the roots of $1 - c_1 z - \ldots - c_p z^p$ are outside the unite circle.

$$(1 - c_1 L - \ldots - c_p L^p)^{-1} = \prod_{j=1}^{p} (1 - \lambda_j L)^{-1}. \tag{7}$$

ARMA

Let $\{\varepsilon_t\}$ be a WN with $Var(\varepsilon_t) = \sigma^2$. An $MA(q)$ process, say $\{X_t\}$ is generated as

$$\begin{aligned}X_t &= \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q} \\ &= \Theta(L) \varepsilon_t,\end{aligned}$$

where

$$\Theta(L) = 1 + \theta_1 L + \ldots + \theta_q L^q.$$

By Theorem 2, $MA(q)$ has the spectral density

$$\begin{aligned}f_X(\lambda) &= \frac{\sigma^2}{2\pi} \left|\Theta\left(e^{-i\lambda}\right)\right|^2 \\ &= \frac{\sigma^2}{2\pi} \left|1 + \theta_1 e^{-i\lambda} + \ldots + \theta_q e^{-i\lambda q}\right|^2,\end{aligned}$$

and the long-run variance

$$\begin{aligned}\omega_X &= 2\pi f_X(0) \\ &= \sigma^2 |\Theta(1)|^2 \\ &= \sigma^2 (1+\theta_1+\ldots+\theta_q)^2.\end{aligned}$$

Notice that for certain values of θ_j's, the long-run variance can be zero. For $0 \leq j \leq q$, the effect of a shock in period t on X after j periods is

$$\theta_j = \frac{\partial X_{t+j}}{\partial \varepsilon_t},$$

and the shocks have no effect after more than q periods.

Definition 1 *(Autoregression) A process $\{X_t\}$ is said to be autoregressive of order p (or autoregression), denoted as $AR(p)$, if it is generated according to*

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a WN.

Let

$$\Phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p.$$

Then, $AR(p)$ can be written as

$$\Phi(L) X_t = \varepsilon_t.$$

Provided that all the roots of $\Phi(z)$ lie outside the unit circle, $\Phi(L)$ can be inverted, and the process has the following $MA(\infty)$ representation (Wold decomposition).

$$\begin{aligned}X_t &= \Phi(L)^{-1}\varepsilon_t \\ &= \Psi(L)\varepsilon_t \\ &= \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}, \qquad (8)\end{aligned}$$

for some $\Psi(L)$.

Suppose that $p = 1$. Then,

$$(1-\phi_1 L) X_t = \varepsilon_t,$$

and, according to (4),

$$\begin{aligned}X_t &= (1-\phi_1 L)^{-1}\varepsilon_t \\ &= \left(\sum_{j=0}^{\infty}\phi_1^j L^j\right)\varepsilon_t \\ &= \sum_{j=0}^{\infty}\phi_1^j\varepsilon_{t-j}.\end{aligned}$$

Hence, in the case of the $AR(1)$ process, the coefficients of $\Psi(L)$ in (8) are given by

$$\psi_j = \phi_1^j.$$

To check the square-summability condition of Theorem 2,

$$\begin{aligned}\sum_{j=0}^{\infty}\psi_j^2 &= \sum_{j=0}^{\infty}\phi_1^{2j} \\ &= \frac{1}{1-\phi_1^2} \\ &< \infty,\end{aligned}$$

provided that

$$|\phi_1| < 1.$$

Then, Theorem 2 implies that $AR(1)$ is covariance stationary, and its spectral density and long-run variance of the $AR(1)$ process are

$$\begin{aligned} f_X(\lambda) &= \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty}\phi_1^j e^{-i\lambda j}\right|^2 = \frac{\sigma^2}{2\pi}\frac{1}{|1-\phi_1 e^{-i\lambda}|^2}, \\ \omega_X &= 2\pi f_X(0) = \frac{\sigma^2}{(1-\phi_1)^2}. \end{aligned}$$

The long-run variance of a stationary $AR(1)$ process is finite as well. Hence, in this case we actually have that the coefficients in $MA(\infty)$ for a stationary $AR(1)$ process are absolutely summable:

$$\sum_{j=0}^{\infty}|\psi_j| = \sum_{j=0}^{\infty}|\phi_1|^j = \frac{1}{1-|\phi_1|} < \infty.$$

This is because $\psi_j = \phi_1^j \to 0$ as $j \to \infty$ at the exponential rate.

In the case of $AR(p)$, $p > 1$, due to (7), $\sum_{j=0}^{\infty}\psi_j^2 < \infty$ provided that all the roots of $\Phi(z)$ lie outside the unit circle. Then $AR(p)$ is covariance stationary with the spectral density and long-run variance

$$\begin{aligned} f_X(\lambda) &= \frac{\sigma^2}{2\pi}\frac{1}{|\Phi(e^{-i\lambda})|^2}, \\ \omega_X &= \frac{\sigma^2}{\Phi(1)^2}. \end{aligned}$$

Definition 2 *(ARMA) A process $\{X_t\}$ is $ARMA(p,q)$, if it is generated according to*

$$\Phi(L)X_t = \Theta(L)\varepsilon_t,$$

where $\{\varepsilon_t\}$ is a WN.

When the roots of $\Phi(z)$ lie outside the unit circle, $ARMA(p,q)$ has an $MA(\infty)$ representation

$$X_t = \Phi(L)^{-1}\Theta(L)\varepsilon_t.$$

Its spectral density and long-run variance are

$$\begin{aligned} f_X(\lambda) &= \frac{\sigma^2}{2\pi}\frac{|\Theta(e^{-i\lambda})|^2}{|\Phi(e^{-i\lambda})|^2}, \\ \omega_X &= \sigma^2\frac{|\Theta(1)|^2}{|\Phi(1)|^2}. \end{aligned}$$

When the roots of $\Theta(z)$ lie outside the unit circle, $ARMA(p,q)$ has an $AR(\infty)$ representation

$$\Theta(L)^{-1}\Phi(L)X_t = \varepsilon_t.$$

Thus, in practice, any covariance stationary $ARMA(p,q)$ or $MA(\infty)$ process can be approximated by $AR(m_n)$ model, with m_n increasing with n, however, at the slower rate.

An ARMA process with the nonzero mean μ can be written as

$$\Phi(L)(X_t-\mu) = \Theta(L)\varepsilon_t.$$

Vector case

Suppose that the vector process $\{X_t\}$ is covariance stationary and $EX_t = 0$ for all t. Let

$$\Gamma(j) = EX_t X_{t-j}'.$$

Notice that, in order for $\{X_t\}$ to be covariance stationary, $\Gamma(j)$ does not need to be symmetric for $j \neq 0$, however,

$$\Gamma(j) = \Gamma(-j)'.$$

Consider a sequence of k-matrices $\{C_j\}$, and define

$$Y_t = \sum_{j=0}^{\infty} C_j X_{t-j}.$$

The variance of Y_t is given by

$$\begin{aligned} EY_tY_t' &= E\left(\sum_{i=0}^{\infty} C_i X_{t-i}\right)\left(\sum_{j=0}^{\infty} C_j X_{t-j}\right)' \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_i \Gamma(j-i) C_j' \\ &= \sum_{j=0}^{\infty} C_j \Gamma(0) C_j' + \sum_{h=1}^{\infty}\sum_{j=0}^{\infty}\left(C_j\Gamma(h) C_{j+h}' + C_{j+h}\Gamma(h)' C_j'\right). \end{aligned}$$

Let $\|A\| = \sqrt{tr(A'A)}$. We have

$$\begin{aligned} \|EY_tY_t'\| &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left\|C_i\Gamma(j-i) C_j'\right\| \\ &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \|C_i\| \|C_j\| \|\Gamma(j-i)\| \\ &\leq 2\sum_{h=0}^{\infty} \|\Gamma(h)\| \left(\sum_{j=0}^{\infty} \|C_j\|^2\right)^2, \end{aligned}$$

where the last inequality is by the same argument as on pages 1-2 of Lecture 9. Hence, EY_tY_t' is finite provided that

$$\sum_{j=0}^{\infty} \|C_j\|^2 < \infty, \tag{9}$$

and

$$\sum_{h=-\infty}^{\infty} \|\Gamma(h)\| < \infty. \tag{10}$$

Definition 3 *A k-vector process $\{\varepsilon_t\}$ is a vector WN if $E\varepsilon_t = 0$ and $E\varepsilon_t\varepsilon_t' = \Sigma$, a positive definite matrix, for all t, and $E\varepsilon_t\varepsilon_s' = 0$ for $t \neq s$.*

If $\{X_t\}$ is purely indeterministic mean zero covariance stationary process such that (10) holds, similarly to the scalar case, it has the $MA(\infty)$ representation

$$X_t = \sum_{j=0}^{\infty} C_j \varepsilon_{t-j},$$

where $\{\varepsilon_t\}$ is a vector WN and C_j's satisfy (9), and

$$C_0 = I_k.$$

Again, as in the scalar case, ε_t's are the linear one step ahead prediction errors.

The spectral density of a vector process is defined similarly to the scalar case:

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\,e^{-i\lambda j} \\ &= \frac{1}{2\pi}\left(\Gamma(0)+\sum_{j=1}^{\infty}\left(\Gamma(j)\,e^{-i\lambda j}+\Gamma(j)'\,e^{i\lambda j}\right)\right), \end{aligned}$$

Again, the long-run variance-covariance matrix is given by

$$\Omega = 2\pi f(0).$$

In order for the long-run variance to be finite,

$$\sum_{j=0}^{\infty}\|C_j\| < \infty.$$

For the vector WN process $\{\varepsilon_t\}$, the spectral density is flat:

$$f_\varepsilon(\lambda) = \frac{1}{2\pi}\Sigma.$$

Let $\{X_t\}$ be a covariance stationary process with the spectral density f_X. Define

$$Y_t = \sum_{j=0}^{\infty} C_j X_{t-j}.$$

Then, the spectral density of $\{Y_t\}$ is given by

$$f_Y(\lambda) = \left(\sum_{j=0}^{\infty} C_j e^{-i\lambda j}\right) f_X(\lambda) \left(\sum_{j=0}^{\infty} C_j' e^{i\lambda j}\right).$$

Let $\{C_j\}$ be a sequence of k-matrices. The lag polynomial $C(L)$ in the vector case is defined as

$$C(L) = I_k + C_1 L + C_2 L^2 + \ldots,$$

where we set $C_0 = I_k$ according to the Wold decomposition. We say

$$B(L) = C(L)^{-1}$$

if

$$B(L)\,C(L) = I_k.$$

The polynomial $C(L)$ is invertible provided that the roots of

$$\det(C(z)) = 0$$

lie outside the unit circle. For example, if $C(L)$ is of order p,

$$\det(I + C_1 z + \ldots + C_p z^p) = 0$$

has to satisfy that all the roots are greater than one in absolute value.