LECTURE 9
LINEAR PROCESSES I: WOLD DECOMPOSITION

In this lecture, we focus on covariance stationary processes.

Definition 1 *(White noise) A process* $\{\varepsilon_t\}$ *is called a white noise (WN) if* $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2 < \infty$ *and* $E\varepsilon_t\varepsilon_{t-j} = 0$ *for all* t *and* $j \neq 0$.

Definition 2 *(Moving average) A process* $\{u_t\}$ *is called the moving average process of order* q *(MA(q)) if*

$$u_t = c_0\varepsilon_t + c_1\varepsilon_{t-1} + \ldots + c_q\varepsilon_{t-q}, \tag{1}$$

and $\{\varepsilon_t\}$ *is a WN.*

A process such as in (1) is called linear. The Wold decomposition says that any mean zero covariance stationary process with absolutely summable autocovariances can be represented in the MA(∞) form.

For a covariance stationary process, we assume that second moments are finite. Let L_2 denote the space of random variables with finite second moments. For $X, Y \in L_2$ define the inner-product

$$\langle X, Y\rangle = EXY.$$

When equipped with such a definition of the inner-product, L_2 is a Hilbert space. Consider the mean zero covariance stationary process $\{X_t\}$, such that

$$\sum_{j=0}^{\infty} |\gamma(j)| < \infty, \tag{2}$$

where

$$\gamma(j) = EX_tX_{t-j}.$$

Define $\mathcal{M}_t$ to be the smallest closed subspace of L_2 that contains all elements of the form

$$\sum_{j=0}^{\infty} c_j X_{t-j} \text{ such that } \sum_{j=0}^{\infty} c_j^2 < \infty.$$

The requirement $\sum_{j=0}^{\infty} c_j^2 < \infty$ is to ensure that the elements of $\mathcal{M}_t$ are in L_2. Indeed, in this case, the variance of any element of $\mathcal{M}_t$ can be bounded using $\sum_{j=0}^{\infty} c_j^2$ and $\sum_{j=0}^{\infty} |\gamma(j)|$:

$$\begin{aligned}
Var\left(\sum_{j=0}^{\infty} c_j X_{t-j}\right) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_i c_j \gamma(i-j) \\
&\leq 2\sum_{h=0}^{\infty}\sum_{j=0}^{\infty} c_j c_{j+h}\gamma(h) \\
&= 2\sum_{h=0}^{\infty}\gamma(h)\sum_{j=0}^{\infty} c_j c_{j+h} \\
&\leq 2\sum_{h=0}^{\infty}|\gamma(h)|\left|\sum_{j=0}^{\infty} c_j c_{j+h}\right| \\
&\leq 2\sum_{h=0}^{\infty}|\gamma(h)|\left(\sum_{j=0}^{\infty} c_j^2\right)^{1/2}\left(\sum_{j=0}^{\infty} c_{j+h}^2\right)^{1/2} \\
&\leq 2\left(\sum_{j=0}^{\infty} c_j^2\right)\sum_{h=0}^{\infty}|\gamma(h)| \\
&< \infty,
\end{aligned}$$

provided that (2) holds. Note further that $\mathcal{M}_t$ is an increasing sequence:

$$\ldots \subset \mathcal{M}_t \subset \mathcal{M}_{t+1} \subset \ldots.$$

Let $P_{\mathcal{M}_t}$ be the orthogonal projection onto $\mathcal{M}_t$. We can write

$$X_t = \widehat{X}_t + \varepsilon_t, \tag{3}$$

where

$$\begin{aligned} \widehat{X}_t &= P_{\mathcal{M}_{t-1}} X_t, \\ \varepsilon_t &= \left(1 - P_{\mathcal{M}_{t-1}}\right) X_t. \end{aligned}$$

From the results for Hilbert spaces, we know that $\widehat{X}_t$ solves the least squares problem. Therefore, $\widehat{X}_t$ can be interpreted as the best one step ahead linear predictor of X_t (in the mean squared error sense), and ε_t is the prediction error. Note that $\widehat{X}_t \in \mathcal{M}_{t-1}$, and $\varepsilon_t \in \mathcal{M}_t$, since

$$\varepsilon_t = X_t - \widehat{X}_t.$$

Further more, by the orthogonal projection result

$$\varepsilon_t \in \mathcal{M}_{t-1}^{\perp},$$

where

$$\mathcal{M}_t^{\perp} = \left\{ Y \in L_2 : \langle Y, X \rangle = 0 \text{ for all } X \in \mathcal{M}_t \right\}.$$

Since $\mathcal{M}_t$ is an increasing sequence, it includes the members of $\mathcal{M}_{t-h}$ and we have that $\varepsilon_t \in \mathcal{M}_{t-h}^{\perp}$ for all $h \geq 1$. Therefore

$$E\varepsilon_t\varepsilon_{t-h} = 0 \text{ for all } h \geq 1.$$

Furthermore,

$$E\varepsilon_t^2 = \sigma^2 < \infty$$

and constant for all t since $\{X_t\}$ is covariance stationary.

Define

$$\mathcal{E}_t = \left\{ \sum_{j=0}^{\infty} c_j \varepsilon_{t-j} : \sum_{j=0}^{\infty} c_j^2 < \infty \right\},$$

where $\mathcal{E}_t$ is actually a closed and linear subspace of L_2. Let $P_{\mathcal{E}_t}$ be an orthogonal projection onto $\mathcal{E}_t$. We have

$$X_t = P_{\mathcal{E}_t} X_t + V_t, \tag{4}$$

where, since $\mathcal{E}_t$ is closed and linear,

$$P_{\mathcal{E}_t} X_t = \sum_{j=0}^{\infty} \widehat{c}_j \varepsilon_{t-j} \tag{5}$$

for some sequence $\{\widehat{c}_j\}$, and

$$V_t = \left(1 - P_{\mathcal{E}_t}\right) X_t.$$

Note that $V_t \in \mathcal{M}_t$, and because of (3),

$$\mathcal{M}_t = \mathcal{M}_{t-1} \oplus \mathcal{S}\left(\varepsilon_t\right),$$

where $\mathcal{S}\left(\varepsilon_t\right)$ is the linear subspace spanned by ε_t, and $\oplus$ denotes the direct sum:

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \left\{ x_1 + x_2 : x_1 \in \mathcal{S}_1, x_2 \in \mathcal{S}_2 \right\}.$$

Since $P_{\mathcal{E}_t}$ is the orthogonal projection, $EV_t\varepsilon_t = 0$ by (4) and (5), and therefore $V_t \notin \mathcal{S}(\varepsilon_t)$. Therefore, it must be true that $V_t \in \mathcal{M}_{t-1}$. By the same argument, since

$$\begin{aligned} \mathcal{M}_{t-1} &= \mathcal{M}_{t-2} \oplus \mathcal{S}(\varepsilon_{t-1}), \text{ and} \\ EV_t\varepsilon_{t-1} &= 0, \end{aligned}$$

we deduce that $V_t \in \mathcal{M}_{t-1}, \mathcal{M}_{t-2}, \ldots$. Let,

$$\mathcal{M}_{-\infty} = \cap_{t=-\infty}^{\infty} \mathcal{M}_t.$$

We conclude that

$$V_t \in \mathcal{M}_{-\infty} \text{ for all } t.$$

Thus, V_t is an element of any linear sub-space $\mathcal{M}_s$, $s \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. The entire process $\{V_t\}$ can be predicted with certainty from an arbitrary distant past of $\{X_t\}$. Such a process is called deterministic.

We derived the Wold representation for a covariance stationary process $\{X_t\}$:

$$X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j} + V_t,$$

where $\{\varepsilon_t\}$ is a WN, and V_t is deterministic. When $V_t = 0$ for all t, the process $\{X_t\}$ is said to be linearly indeterministic. If $\{X_t\}$ is indeterministic and mean zero, it has the MA(∞) representation

$$X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}.$$

One can show that

$$\begin{aligned} c_j &= EX_t\varepsilon_{t-j}/\sigma^2, \text{ and} && (6) \\ c_0 &= 1. && (7) \end{aligned}$$

Remarks:

1. The representation is unique with probability one, which follows from uniqueness of the Hilbert space projection.

2. The WN elements in the Wold decomposition are the one step ahead linear prediction errors.

3. Sometimes ε_t's are interpreted as the fundamental shocks of the economy. Then the impulse responses, c_j's, represent the effect of a shock after j periods. Suppose that the fundamental shocks are given by the unexpected shocks to the agents' information set. Let $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$. $\mathcal{F}_t$ represents the information of the agents in the economy at time t. Note that $\widehat{X}_t$ is restricted to be a linear predictor, and, therefore, is not necessary equal to $E(X_t|\mathcal{F}_t)$. Hence, the true fundamental shocks may differ from the shocks that enter the Wold representation (see, for example Hansen and Sargent (1991), Chapter 4).