LECTURE 5
SIMULTANEOUS EQUATIONS IV: LIMITED INFORMATION ML (LIML)

In this lecture, we consider ML estimation of a single equation which is a part of the system of simultaneous equations. Without loss of generality, we can focus on the first equation:

$$\begin{aligned} y_{1i} &= X_{1,i}'\delta_1 + u_{1i} \\ &= Y_{1,i}'\gamma_1 + Z_{1,i}'\beta_1 + u_{1i}, \end{aligned}$$

where $Y_{1,i}$ and $Z_{1,i}$ are the vectors of included endogenous and exogenous regressors respectively, as defined in Lecture 2. For the included endogenous regressors we have the following reduced form equation

$$Y_{1,i} = \Pi_1 Z_{1,i} + \Pi_2 Z_{2,i} + V_{1,i}.$$

Note that we ignore $Y_{1,i}^*$, the vector of endogenous variables excluded from the first equation. The two above equations can be written together as

$$\begin{pmatrix} 1 & -\gamma_1' \\ 0 & I_{m_1} \end{pmatrix} \begin{pmatrix} y_{1i} \\ Y_{1,i} \end{pmatrix} = \begin{pmatrix} \beta_1' & 0 \\ \Pi_1 & \Pi_2 \end{pmatrix} \begin{pmatrix} Z_{1,i} \\ Z_{2,i} \end{pmatrix} + \begin{pmatrix} u_{1i} \\ V_{1,i} \end{pmatrix},$$

or

$$\widetilde{\Gamma}_1 \widetilde{Y}_{1,i} = \widetilde{B}_1 Z_i + \widetilde{U}_i,$$

where

$$\begin{aligned} \widetilde{\Gamma}_1 &= \begin{pmatrix} 1 & -\gamma_1' \\ 0 & I_{m_1} \end{pmatrix}, \\ \widetilde{B}_1 &= \begin{pmatrix} \beta_1' & 0 \\ \Pi_1 & \Pi_2 \end{pmatrix}, \\ \widetilde{Y}_{1,i} &= \begin{pmatrix} y_{1i} \\ Y_{1,i} \end{pmatrix}, \\ \widetilde{U}_i &= \begin{pmatrix} u_{1i} \\ V_{1,i} \end{pmatrix}. \end{aligned}$$

Assuming that

$$\widetilde{U}_i | Z_i \sim N\left(0, \widetilde{\Sigma}_1\right),$$

similarly to the derivation of equation (3) in Lecture 4, we obtain that the concentrated log-likelihood for $\widetilde{Y}_{1,i}$ is

$$\begin{aligned} Q_n\left(\widetilde{\Gamma}_1, \widetilde{B}_1\right) &= -\frac{(m_1+1)}{2}\left(\log(2\pi)+1\right) + \log\left|\widetilde{\Gamma}_1\right| - \frac{1}{2}\log\left| n^{-1}\sum_{i=1}^n \left(\widetilde{\Gamma}_1\widetilde{Y}_{1,i} - \widetilde{B}_1 Z_i\right)\left(\widetilde{\Gamma}_1\widetilde{Y}_{1,i} - \widetilde{B}_1 Z_i\right)'\right| \\ &= -\frac{(m_1+1)}{2}\left(\log(2\pi)+1\right) - \frac{1}{2}\log\left| n^{-1}\sum_{i=1}^n \left(\widetilde{\Gamma}_1\widetilde{Y}_{1,i} - \widetilde{B}_1 Z_i\right)\left(\widetilde{\Gamma}_1\widetilde{Y}_{1,i} - \widetilde{B}_1 Z_i\right)'\right|, \end{aligned}$$

where the last equality follows from the fact that due to restricted structure of $\widetilde{\Gamma}_1$, $\left|\widetilde{\Gamma}_1\right| = 1$.

Thus, the LIML is a special case of FIML with properly defined matrices of parameters. However, again, due to the restricted structure of $\widetilde{\Gamma}_1$, there exists a closed form expression for the LIML estimator. Let $\widehat{\delta}_1$ be the LIML estimator of $\delta_1 = \left(\gamma_1', \beta_1'\right)'$, then using the matrix notation of Lecture 2, we can write

$$\widehat{\delta}_1 = \left(X_1'\left(I_n - \lambda M\right)X_1\right)^{-1} X_1'\left(I_n - \lambda M\right) y_1,$$

where

$$\begin{aligned} M &= I_n - P, \\ P &= Z\left(Z'Z\right)^{-1} Z', \end{aligned}$$

and P is the projection matrix onto the space spanned by the exogenous variables Z_i's (included and excluded from the first equation),

$$\lambda = \min_t \frac{t'W_1 t}{t'Wt},$$

where

$$\begin{aligned} W &= \begin{pmatrix} y_1 & Y_1 \end{pmatrix}' M \begin{pmatrix} y_1 & Y_1 \end{pmatrix}, \\ W_1 &= \begin{pmatrix} y_1 & Y_1 \end{pmatrix}' M_1 \begin{pmatrix} y_1 & Y_1 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} M_1 &= I_n - P_1, \\ P_1 &= Z_1\left(Z_1'Z_1\right)^{-1} Z_1', \end{aligned}$$

and M_1 projection matrix onto space orthogonal to that spanned by $Z_{1,i}$'s, the exogenous variables included in the first equation. (As defined above, λ is actually the smallest eigenvalue of $W_1 W^{-1}$.)

Next, we will show the asymptotic equivalence of LIML and 2SLS estimators. First, we will show that $\lambda \geq 1$.

$$\begin{aligned} t'W_1 t - t'Wt &= t' \begin{pmatrix} y_1 & Y_1 \end{pmatrix}' (M_1 - M) \begin{pmatrix} y_1 & Y_1 \end{pmatrix} t \\ &= t' \begin{pmatrix} y_1 & Y_1 \end{pmatrix}' (P - P_1) \begin{pmatrix} y_1 & Y_1 \end{pmatrix} t. \end{aligned}$$

Since Z_1 is a part of Z, $PZ_1 = Z_1$, and, therefore, $PP_1 = P_1$. Hence,

$$\begin{aligned} (P - P_1)(P - P_1) &= P - P_1 P - PP_1 + P_1 \\ &= P - P_1, \end{aligned}$$

idempotent and, therefore, positive definite. Thus, $t'W_1 t - t'Wt \geq 0$ for any t and $\lambda \geq 1$.

Next, define

$$u_1 = \begin{pmatrix} u_{1i} \\ \vdots \\ u_{1n} \end{pmatrix}.$$

We have

$$\begin{aligned} & \min_t \frac{t'W_1 t}{t'Wt} \\ \leq \; & \frac{\begin{pmatrix} 1 & -\gamma_1' \end{pmatrix} W_1 \begin{pmatrix} 1 & -\gamma_1' \end{pmatrix}'}{\begin{pmatrix} 1 & -\gamma_1' \end{pmatrix} W \begin{pmatrix} 1 & -\gamma_1' \end{pmatrix}'} \\ \leq \; & \frac{\begin{pmatrix} 1 & -\gamma_1' \end{pmatrix} \begin{pmatrix} y_1 & Y_1 \end{pmatrix}' M_1 \begin{pmatrix} y_1 & Y_1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma_1' \end{pmatrix}'}{\begin{pmatrix} 1 & -\gamma_1' \end{pmatrix} \begin{pmatrix} y_1 & Y_1 \end{pmatrix}' M \begin{pmatrix} y_1 & Y_1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma_1' \end{pmatrix}'} \\ = \; & \frac{\left(Z_1\beta_1 + u_1\right)' M_1 \left(Z_1\beta_1 + u_1\right)}{\left(Z_1\beta_1 + u_1\right)' M \left(Z_1\beta_1 + u_1\right)} \\ = \; & \frac{u_1' M_1 u_1}{u_1' M u_1}. \end{aligned}$$

Thus,

$$0 \leq \lambda - 1 \leq \frac{u_1' \left(M_1 - M\right) u_1}{u_1' M u_1} = \frac{u_1' \left(P - P_1\right) u_1}{u_1' M u_1}.$$

Lastly,

$$\begin{aligned} n^{1/2} \frac{u_1' \left(P - P_1\right) u_1}{u_1' M u_1} &= \frac{\frac{u_1' Z}{n^{1/2}} \left(\frac{Z'Z}{n}\right)^{-1} \frac{Z' u_1}{n} - \frac{u_1' Z_1}{n^{1/2}} \left(\frac{Z_1' Z_1}{n}\right)^{-1} \frac{Z_1' u_1}{n}}{\frac{u_1' u_1}{n} - \frac{u_1' Z}{n} \left(\frac{Z'Z}{n}\right)^{-1} \frac{Z' u_1}{n}} \\ &\to_p 0, \end{aligned}$$

and, therefore,

$$n^{1/2} \left(\lambda - 1\right) \to_p 0.$$

Next,

$$\begin{aligned} I_n - \lambda M &= I_n - \lambda M + M - M \\ &= I_n - M - \left(\lambda - 1\right) M \\ &= P - \left(\lambda - 1\right) M. \end{aligned}$$

Hence, the difference between the LIML and 2SLS estimators is given by

$$\begin{aligned} n^{1/2} \left(\widehat{\delta}_1 - \widehat{\delta}_1^{2SLS}\right) &= \left(\frac{X_1' \left(I_n - \lambda M\right) X_1}{n}\right)^{-1} \frac{X_1' \left(I_n - \lambda M\right) u_1}{n^{1/2}} - \left(\frac{X_1' P X_1}{n}\right)^{-1} \frac{X_1' P u_1}{n^{1/2}} \\ &= \left(\frac{X_1' P X_1 - \left(\lambda - 1\right) X_1' M X_1}{n}\right)^{-1} \frac{X_1' P u_1 - \left(\lambda - 1\right) X_1' M u_1}{n^{1/2}} - \left(\frac{X_1' P X_1}{n}\right)^{-1} \frac{X_1' P u_1}{n^{1/2}} \\ &= \left(\left(\frac{X_1' P X_1 - \left(\lambda - 1\right) X_1' M X_1}{n}\right)^{-1} - \left(\frac{X_1' P X_1}{n}\right)^{-1}\right) \frac{X_1' P u_1}{n^{1/2}} \\ &- \left(\frac{X_1' P X_1 - \left(\lambda - 1\right) X_1' M X_1}{n}\right)^{-1} \frac{\left(\lambda - 1\right) X_1' M u_1}{n^{1/2}} \\ &\to_p 0. \end{aligned}$$