LECTURE 2

SIMULTANEOUS EQUATIONS I: DEFINITION, IDENTIFICATION, INDIRECT LS, SINGLE-EQUATION GMM

Definition

We consider the following *system of equations*:

$$\begin{aligned} \Gamma Y_i &= BZ_i + U_i, && (1) \\ EZ_iU_i' &= 0, && (2) \end{aligned}$$

where Y_i is an m-vector of *endogenous* variables:

$$Y_i = \begin{pmatrix} y_{1i} \\ \vdots \\ y_{mi} \end{pmatrix},$$

Z_i is an l-vector of *exogenous* variables, U_i is an m-vector of residuals:

$$U_i = \begin{pmatrix} u_{1i} \\ \vdots \\ u_{mi} \end{pmatrix}.$$

The unknown parameters are given by the $m \times m$ matrix Γ and $m \times l$ matrix B. The econometrician observes the data $\{(Y_i', Z_i') : i = 1, \ldots, n\}$.

We have m equations determining m endogenous variables. The first equation is given by

$$\sum_{j=1}^{m} \Gamma_{1j} y_{ji} = \sum_{j=1}^{l} B_{1j} Z_{ji} + u_{1i}.$$

Multiplying all Γ_{1j}'s and B_{1j}'s by a nonzero constant would affect the variance of unobservable error term u_{1i}., however, it would not change the relationship between the observable variables, i.e. Γ and B can be identified only up to scale. hence, we must introduce some normalization to the system. We will assume that $\Gamma_{jj} = 1$ for all $j = 1, \ldots, m$. Later, this will allow us to treat y_{ij} as the "dependent" variable in equation j.

Next, we assume that Γ^{-1} exists, since, otherwise, there is no unique expression for Y_i in terms of exogenous variables Z's and shocks U's. Let's rewrite (1) as

$$\begin{aligned} Y_i &= \Gamma^{-1}BZ_i + \Gamma^{-1}U_i, \text{ or} \\ Y_i &= \Pi Z_i + V_i, \text{ where} && (3) \\ \Pi &= \Gamma^{-1}B, \\ V_i &= \Gamma^{-1}U_i. \end{aligned}$$

The system of equations in (3) is called the *reduced form* as opposed to *structural* equations in (1). Due to (2) we have that $EZ_iV_i' = 0$, and, consequently, the matrix of the reduced form parameters Π can be estimated consistently by OLS. The reduced form is useful, for example, for forecasting of Y's from Z's, however, we are usually interested in estimating economic relations that are given by the structural equations. The structural parameters cannot be estimated directly by OLS, since each of the structural equations has Y's on the right-hand side, and in general,

$$\begin{aligned} EY_iU_i' &= \Pi EZ_iU_i' + \Gamma^{-1}EU_iU_i' \\ &= \Gamma^{-1}Var(U_i) \\ &\neq 0. \end{aligned}$$

Hence, the first question is whether it is possible to recover the structural parameters Γ from Π.

Structural equation is said to be *identified* if its coefficients are *uniquely* determined by the elements of Π. The structural equation is not identified when the structural parameters cannot be recovered *uniquely* from Π (we know that there is a solution for Γ and B because by construction $\Pi = \Gamma^{-1}B$). Simple counting shows that there are $m^2 - m + ml$ unknown parameters in Γ and B, while there are only ml reduced form coefficients in Π. Thus, without imposing additional restrictions on the structural parameters, the system or its parts cannot be identified. The most common type of restrictions is a zero restriction or exclusion of variables from equations. It is possible that some equations are identified while others are not, and, therefore, identification should be considered equation by equation.

Without loss of generality, let's consider the first equation. Suppose that the first equation has m_1 endogenous and l_1 exogenous variables *included* on the right-hand side. Write

$$\Gamma = \begin{pmatrix} 1 & -\gamma_1' & 0' \\ \Gamma_1 & \Gamma_2 & \Gamma_3 \end{pmatrix},$$

where, in the first row, γ_1 is an m_1-vector with all elements different from zero, and 0 is actually an $(m - m_1 - 1)$-vector of zeros;

$$B = \begin{pmatrix} \beta_1' & 0' \\ B_1 & B_2 \end{pmatrix},$$

where in the first row β_1 is an l_1-vector with all elements different from zero, and 0 is an $(l - l_1)$-vector of zeros. Accordingly, let's partition Π such that $\Gamma\Pi = B$:

$$\begin{pmatrix} 1 & -\gamma_1' & 0' \\ \Gamma_1 & \Gamma_2 & \Gamma_3 \end{pmatrix} \begin{pmatrix} \pi_1 & \pi_2 \\ \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{pmatrix} = \begin{pmatrix} \beta_1' & 0' \\ B_1 & B_2 \end{pmatrix}, \text{ and}$$

$$\pi_1 - \gamma_1'\Pi_1 = \beta_1', \tag{4}$$

$$\pi_2 - \gamma_1'\Pi_2 = 0', \tag{5}$$

where π_1 is $1 \times l_1$, π_2 is $1 \times (l - l_1)$, Π_1 is $m_1 \times l_1$, and Π_2 is $m_1 \times (l - l_1)$. The expression in (4) says that we can find β_1 if we know γ_1. Next, (5) is a system of $l - l_1$ equations with m_1 unknowns (γ_1). In order to be able to solve for γ_1, there should be at least as many equations as unknowns, which gives us the following *order condition*:

$$l - l_1 \geq m_1,$$

i.e. the number of exogenous variables excluded from the equation should exceed the number of endogenous variables included on the right-hand side. The order condition is only necessary but not sufficient. Given the fact that there is a solution (because Π is determined by Γ and B), the necessary and sufficient condition for there to be a unique solution for γ_1 is the following *rank condition*:

$$\text{rank}(\Pi_2) = m_1$$

(see the appendix for a proof). Thus, the equation is *not identified* when $l - l_1 < m_1$ or the rank condition fails. It is *overidentified* when $l - l_1 > m_1$, and the rank condition is met. The equations is *exactly identified* when $l - l_1 = m_1$, and the rank condition is met In the exactly identified case, we have

$$\gamma_1' = \pi_2\Pi_2^{-1}. \tag{6}$$

Indirect LS

Since Π can be estimated consistently by OLS, we can use (6) in order to estimate the structural parameters of an exactly identified equation. The indirect LS estimator of γ_1 is given by

$$\widehat{\gamma}_1^{ILS} = \left(\widehat{\Pi}_2'\right)^{-1} \widehat{\pi}_2',$$

where $\widehat{\pi}_2$ and $\widehat{\Pi}_2$ are the LS estimators of π_2 and Π_2 respectively.

Let's write

$$Y_i = \left(\begin{array}{ccc} y_{1i} & Y_{1,i}' & Y_{1,i}^{*\prime} \end{array} \right)',$$

where Y_{1i} is the m_1-vector of the endogenous variables included on the right-hand side in equation 1, and Y_{1i}^* is the $(m - m_1 - 1)$-vector of excluded endogenous variables. Similarly, write

$$Z_i = \left(\begin{array}{cc} Z_{1,i}' & Z_{1,i}^{*\prime} \end{array} \right)',$$

where $Z_{1,i}$ and $Z_{1,i}^*$ are the vectors of included and excluded exogenous variables respectively (for the first equation). Further, write

$$\left(\begin{array}{c} y_{1i} \\ Y_{1,i} \\ Y_{1,i}^* \end{array} \right) = \left(\begin{array}{cc} \pi_1 & \pi_2 \\ \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{array} \right) \left(\begin{array}{c} Z_{1,i} \\ Z_{1,i}^* \end{array} \right) + \left(\begin{array}{c} v_{1i} \\ V_{1,i} \\ V_{1,i}^* \end{array} \right).$$

Note that π_2 is the vector of reduced form coefficients of Z_{1i}^* in the y_1's equation; and Π_2 gives the coefficients of Z_{1i}^* in the Y_1's equation. We have

$$\begin{aligned} \widehat{\pi}_2' &= \left(Z_1^{*\prime} M_1 Z_1^*\right)^{-1} Z_1^{*\prime} M_1 y_1, \\ \widehat{\Pi}_2' &= \left(Z_1^{*\prime} M_1 Z_1^*\right)^{-1} Z_1^{*\prime} M_1 Y_1, \end{aligned}$$

where

$$\begin{aligned} y_1 &= \left(\begin{array}{c} y_{11} \\ \vdots \\ y_{1n} \end{array} \right), \\ Y_1 &= \left(\begin{array}{c} Y_{1,i}' \\ \vdots \\ Y_{1,n}' \end{array} \right), \\ Z_1^* &= \left(\begin{array}{c} Z_{1,1}^{*\prime} \\ \vdots \\ Z_{1,n}^{*\prime} \end{array} \right), \\ Z_1 &= \left(\begin{array}{c} Z_{1,1}' \\ \vdots \\ Z_{1,n}' \end{array} \right), \\ M_1 &= I_n - Z_1 \left(Z_1' Z_1\right)^{-1} Z_1'. \end{aligned}$$

Hence,

$$\widehat{\gamma}_1^{ILS} = \left(Z_1^{*\prime} M_1 Y_1\right)^{-1} Z_1^{*\prime} M_1 y_1.$$

Note that this is just an IV estimator of Y_1 from the regression of y_1 against Y_1 and Z_1, where Z_1^* were used as instruments for Y_1. Indeed, write the IV estimated equation as

$$y_1 = Y_1 \widehat{\gamma}_1^{IV} + Z_1 \widehat{\beta}_1^{IV} + \widehat{u}_1,$$

where $Z_1' \widehat{u}_1 = 0$ and $Z_1^{*\prime} \widehat{u}_1 = 0$. Then,

$$\begin{aligned} \widehat{\gamma}_1^{ILS} &= \left(Z_1^{*\prime} M_1 Y_1\right)^{-1} Z_1^{*\prime} M_1 \left(Y_1 \widehat{\gamma}_1^{IV} + Z_1 \widehat{\beta}_1^{IV} + \widehat{u}_1 \right) \\ &= \widehat{\gamma}_1^{IV} + \left(Z_1^{*\prime} M_1 Y_1\right)^{-1} Z_1^{*\prime} M_1 \widehat{u}_1, \end{aligned}$$

and

$$\begin{aligned} Z_1^{*\prime} M_1 \widehat{u}_1 &= Z_1^{*\prime} \widehat{u}_1 - Z_1^{*\prime} Z_1 \left(Z_1^{\prime} Z_1\right)^{-1} Z_1^{\prime} \widehat{u}_1 \\ &= 0. \end{aligned}$$

The Indirect LS estimator is unique only if the equation is exactly identified. In case of an overidentified equation, one could choose some m_1 restrictions out of $l - l_1$ in (5), in order to construct the indirect LS estimator. However, a better approach is to use GMM estimation.

Identification and GMM estimation of a single equation

Consider the case where the first equation is identified, i.e. $l - l_1 \geq m_1$, and Π_2 has the rank m_1 (the rank condition is satisfied). Let's re-write the first equation again as

$$\begin{aligned} y_{1i} &= Y_{1,i}^{\prime} \gamma_1 + Z_{1,i}^{\prime} \beta_1 + u_{1i} \\ &= X_{1,i}^{\prime} \delta_1 + u_{1i}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} X_{1,i} &= \begin{pmatrix} Y_{1,i}^{\prime} & Z_{1,i}^{\prime} \end{pmatrix}^{\prime}, \\ \delta_1 &= \begin{pmatrix} \gamma_1^{\prime} & \beta_1^{\prime} \end{pmatrix}^{\prime}. \end{aligned}$$

The equation can be estimated by GMM provided that the rank condition for GMM estimation is satisfied, i.e. $\operatorname{rank}\left(E Z_i X_{1,i}^{\prime}\right) = m_1 + l_1$. For this to hold, the necessary condition is that the number of instruments in Z_i must exceed the number of regressors in X_i: $l \geq m_1 + l_1$ or $l - l_1 \geq m_1$, which gives us back the order condition. Next, we will see that $\operatorname{rank}\left(E Z_i X_{1,i}^{\prime}\right) = m_1 + l_1$ is equivalent to $\operatorname{rank}(\Pi_2) = m_1$.

$$\begin{aligned} E Z_i X_{1,i}^{\prime} &= E \begin{pmatrix} Z_{1,i} \\ Z_{1,i}^{*} \end{pmatrix} \begin{pmatrix} Y_{1,i}^{\prime} & Z_{1,i}^{\prime} \end{pmatrix} \\ &= E \begin{pmatrix} Z_{1,i} \\ Z_{1,i}^{*} \end{pmatrix} \begin{pmatrix} \left(\Pi_1 Z_{1,i} + \Pi_2 Z_{1,i}^{*} + V_{1,i}\right)^{\prime} & Z_{1,i}^{\prime} \end{pmatrix} \\ &= \begin{pmatrix} E\left(Z_{1,i} Z_{1,i}^{\prime}\right) \Pi_1^{\prime} + E\left(Z_{1,i} Z_{1,i}^{*\prime}\right) \Pi_2^{\prime} & E\left(Z_{1,i} Z_{1,i}^{\prime}\right) \\ E\left(Z_{1,i}^{*} Z_{1,i}^{\prime}\right) \Pi_1^{\prime} + E\left(Z_{1,i}^{*} Z_{1,i}^{*\prime}\right) \Pi_2^{\prime} & E\left(Z_{1,i}^{*} Z_{1,i}^{\prime}\right) \end{pmatrix}. \end{aligned}$$

Suppose this $l \times (m_1 + l_1)$ matrix does not have the full column rank. This is the case if and only if there exists some nonzero $(m_1 + l_1)$-vector $\theta = \begin{pmatrix} \theta_1^{\prime} & \theta_2^{\prime} \end{pmatrix}^{\prime}$ such that $E Z_i X_{1,i}^{\prime} \theta = 0$:

$$\begin{aligned} 0 &= \begin{pmatrix} E\left(Z_{1,i} Z_{1,i}^{\prime}\right) \Pi_1^{\prime} + E\left(Z_{1,i} Z_{1,i}^{*\prime}\right) \Pi_2^{\prime} & E\left(Z_{1,i} Z_{1,i}^{\prime}\right) \\ E\left(Z_{1,i}^{*} Z_{1,i}^{\prime}\right) \Pi_1^{\prime} + E\left(Z_{1,i}^{*} Z_{1,i}^{*\prime}\right) \Pi_2^{\prime} & E\left(Z_{1,i}^{*} Z_{1,i}^{\prime}\right) \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\ &= \begin{pmatrix} E\left(Z_{1,i} Z_{1,i}^{\prime}\right) \Pi_1^{\prime} \theta_1 + E\left(Z_{1,i} Z_{1,i}^{*\prime}\right) \Pi_2^{\prime} \theta_1 + E\left(Z_{1,i} Z_{1,i}^{\prime}\right) \theta_2 \\ E\left(Z_{1,i}^{*} Z_{1,i}^{\prime}\right) \Pi_1^{\prime} \theta_1 + E\left(Z_{1,i}^{*} Z_{1,i}^{*\prime}\right) \Pi_2^{\prime} \theta_1 + E\left(Z_{1,i}^{*} Z_{1,i}^{\prime}\right) \theta_2 \end{pmatrix} \\ &= \begin{pmatrix} E\left(Z_{1,i} Z_{1,i}^{\prime}\right) & E\left(Z_{1,i} Z_{1,i}^{*\prime}\right) \\ E\left(Z_{1,i}^{*} Z_{1,i}^{\prime}\right) & E\left(Z_{1,i}^{*} Z_{1,i}^{*\prime}\right) \end{pmatrix} \begin{pmatrix} \Pi_1^{\prime} \theta_1 + \theta_2 \\ \Pi_2^{\prime} \theta_1 \end{pmatrix}. \end{aligned}$$

Assuming that $E Z_i Z_i^{\prime}$ is positive definite, $E Z_i X_{1,i}^{\prime}$ does not have the full rank if and only if

$$\begin{aligned} \Pi_1^{\prime} \theta_1 + \theta_2 &= 0, \\ \Pi_2^{\prime} \theta_1 &= 0, \end{aligned} \tag{8}$$

i.e. the m_1 columns of $\Pi_2^{\prime}$ are linearly dependent. (Note that both θ_1 and θ_2 must be non-zero, since if $\theta_1 = 0$, then $0 = \Pi_1^{\prime} \theta_1 + \theta_2 = 0 + \theta_2 = \theta_2$.) Hence, $\operatorname{rank}\left(E Z_i X_{1,i}^{\prime}\right) < m_1 + l_1$ implies that $\operatorname{rank}(\Pi_2) < m_1$. Consequently, $\operatorname{rank}(\Pi_2) = m_1$ implies that $\operatorname{rank}\left(E Z_i X_{1,i}^{\prime}\right) = m_1 + l_1$.

Next, for any $\theta = (\theta_1', \theta_2')'$ we have that

$$\left(EZ_iX_{1,i}'\right)\theta = \begin{pmatrix} E\left(Z_{1,i}Z_{1,i}'\right) & E\left(Z_{1,i}Z_{1,i}^{*\prime}\right) \\ E\left(Z_{1,i}^{*}Z_{1,i}'\right) & E\left(Z_{1,i}^{*}Z_{1,i}^{*\prime}\right) \end{pmatrix} \begin{pmatrix} \Pi_1'\theta_1 + \theta_2 \\ \Pi_2'\theta_1 \end{pmatrix}$$

Suppose that $\operatorname{rank}(\Pi_2) < m_1$. Then, there exists a nonzero θ_1 such that (8) holds. Next, define $\theta_2 = -\Pi_1'\theta_1$. For such a choice of $\theta = (\theta_1', \theta_2')'$ we have that $\left(EZ_iX_{1,i}'\right)\theta = 0$. Hence, $\operatorname{rank}(\Pi_2) < m_1$ implies that $\operatorname{rank}\left(EZ_iX_{1,i}'\right) < m_1 + l_1$, and, therefore, $\operatorname{rank}\left(EZ_iX_{1,i}'\right) = m_1 + l_1$ implies $\operatorname{rank}(\Pi_2) = m_1$. Thus, the two rank conditions are equivalent: $EZ_iX_{1,i}'$ has the full column rank $m_1 + l_1$ if and only if the rank of Π_2 is m_1.

We have shown that the GMM rank condition on $EZ_iX_{1,i}'$ is equivalent to the rank condition on the corresponding matrix of the reduced form parameters. Thus, provided that the equation is identified, one can use the usual GMM technique to estimate that equation:

$$\widetilde{\delta}_1\left(A_{1n}\right) = \left(\sum_{i=1}^{n} X_{1,i}Z_i'\left(A_{1n}'A_{1n}\right)\sum_{i=1}^{n} Z_iX_{1,i}'\right)^{-1} \sum_{i=1}^{n} X_{1,i}Z_i'\left(A_{1n}'A_{1n}\right)\sum_{i=1}^{n} Z_iy_{1i}.$$

The efficient A_{1n} is such that $A_{1n}'A_{1n} \to_p \left(Eu_{1i}^2Z_iZ_i'\right)^{-1}$. The efficient GMM reduces to the 2SLS estimator in the homoskedastic case ($A_{1n}'A_{1n} = (\sum_{i=1}^{n} Z_iZ_i')^{-1}$), which in turn is the same as the IV estimator when the system is exactly identified.

The estimators of δ_j are asymptotically correlated across the equations, even if different equations in the system are estimated separately. Suppose that the first two equations are identified. We have

$$n^{1/2}\begin{pmatrix} \widetilde{\delta}_1\left(A_{1n}\right) - \delta_1 \\ \widetilde{\delta}_2\left(A_{2n}\right) - \delta_2 \end{pmatrix} = \begin{pmatrix} S_{1,n} & 0 \\ 0 & S_{2,n} \end{pmatrix} n^{-1/2}\sum_{i=1}^{n}\begin{pmatrix} Z_iu_{1i} \\ Z_iu_{2i} \end{pmatrix},$$

where

$$S_{j,n}\left(A_{jn}\right) = \left(n^{-1}\sum_{i=1}^{n} X_{j,i}Z_i'\left(A_{jn}'A_{jn}\right)n^{-1}\sum_{i=1}^{n} Z_iX_{j,i}'\right)^{-1} n^{-1}\sum_{i=1}^{n} X_{j,i}Z_i'\left(A_{jn}'A_{jn}\right),$$

for $j = 1, 2$. Next, under the usual assumptions

$$\begin{aligned} S_{j,n}\left(A_{jn}\right) &\to_p S_j\left(A_j\right) \\ &= \left(Q_j'A_j'A_jQ_j\right)^{-1} Q_j'A_j'A_j, \end{aligned}$$

where $Q_j = EZ_iX_{j,i}'$. Further,

$$\begin{aligned} & n^{-1/2}\sum_{i=1}^{n}\begin{pmatrix} Z_iu_{1i} \\ Z_iu_{2i} \end{pmatrix} \\ = \quad & n^{-1/2}\sum_{i=1}^{n}\begin{pmatrix} u_{1i} \\ u_{2i} \end{pmatrix} \otimes Z_i \to_d N\left(0, \Omega_{1,2}\right), \end{aligned}$$

where $A \otimes B$ denotes the Kronecker product of A and B,[1] and

$$\Omega_{1,2} = E\left(\begin{pmatrix} u_{1i}^2 & u_{1i}u_{2i} \\ u_{1i}u_{2i} & u_{2i}^2 \end{pmatrix} \otimes Z_iZ_i'\right).$$

[1]Suppose that A is $k \times l$ and B is $m \times n$. Then $A \otimes B$ is a $(km) \times (ln)$ matrix given by

$$A \otimes B = \begin{pmatrix} a_{11} & \dots & a_{1l} \\ & \dots & \\ a_{k1} & \dots & a_{kl} \end{pmatrix} \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1l}B \\ & \dots & \\ a_{k1}B & \dots & a_{kl}B \end{pmatrix}.$$

The properties of the Kronecker product include: $(A \otimes B)' = A' \otimes B'$, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ when A and B are invertible, and $(A \otimes B)(C \otimes D) = AC \otimes BD$ for properly defined matrices C and D.

Thus,

$$n^{1/2}\begin{pmatrix}\widetilde{\delta}_1(A_{1n})-\delta_1\\ \widetilde{\delta}_2(A_{2n})-\delta_2\end{pmatrix}\to_d N\left(0,V\left(A_1,A_2\right)\right),$$

where

$$V\left(A_1,A_2\right)=\begin{pmatrix}S_1(A_1) & 0\\ 0 & S_2(A_2)\end{pmatrix}\Omega_{1,2}\begin{pmatrix}S_1(A_1) & 0\\ 0 & S_2(A_2)\end{pmatrix}'.$$

One can see that the asymptotic covariance between $\widetilde{\delta}_1(A_{1n})$ and $\widetilde{\delta}_2(A_{2n})$ is given by

$$\left(Q_1'A_1'A_1Q_1\right)^{-1}Q_1'A_1'A_1E\left(u_{1i}u_{2i}Z_iZ_i'\right)A_2'A_2Q_2\left(Q_2'A_2'A_2Q_2\right)^{-1}$$

Appendix: solving a linear system of equations

This discussion follows Magnus J.R. and H. Neudecker (2007): "*Matrix Differential Calculus.*"

Consider a system of linear equations

$$Ax=b,$$

where A is a known $l\times m$ matrix and b is a known l-vector. We are looking for an m-vector x that solves the system. In the case of a system of simultaneous equations, $A=\Pi_2', x=\gamma_1$, and $b=\pi_2'$.

A solution can be conveniently expresses using a generalized inverse of A. A generalized inverse of A is a matrix A^- such that $AA^-A=A$. A generalized inverse is not unique. A Moore-Penrose inverse of a matrix A (denoted by A^+) is a generalized inverse that satisfies the following conditions:

1. $AA^+A=A$.
2. $A^+AA^+=A^+$.
3. A^+A is symmetric.
4. AA^+ is symmetric.

A unique A^+ always exists for any matrix A. For example, if A has full column rank, i.e. $rank\left(A\right)=m$, then

$$A^+=\left(A'A\right)^{-1}A',$$

which can be easily checked by verifying conditions 1-4.

We will establish below several results useful in characterizing a solution to $Ax=b$. First, let's consider a *homogeneous* system of linear equations

$$Ax=0$$

(note that in this case there is always a solution $x=0$).

Lemma 1 *A general solution to the homogeneous system of linear equations $Ax=0$ is*

$$x=\left(I_m-A^+A\right)q,$$

where q is an arbitrary m-vector.

Proof. Obviously, $x=\left(I_m-A^+A\right)q$ is a solutions since, due to property 1 of a Moore-Penrose inverse,

$$\begin{aligned}A\left(I_m-A^+A\right)q &= \left(A-AA^+A\right)q\\ &= \left(A-A\right)q\\ &= 0.\end{aligned}$$

Next, suppose x is a solution. Then, it has to satisfy $x=\left(I_m-A^+A\right)x$, because $\left(I_m-A^+A\right)x=x-A^+Ax=x-A^+0=x$. Thus, any solution can be written as $x=\left(I_m-A^+A\right)q$ for some q. ■

Lemma 2 *$Ax = b$ has a solution if and only if rank $(A) =$ rank $\left(\left[\begin{array}{cc} A & b \end{array}\right]\right)$.*

Proof. Write $A = \left[\begin{array}{ccc} a_1 & \ldots & a_m \end{array}\right]$, where a_i is an l-vector, $i = 1, \ldots, m$. Suppose the vector $x = (x_1, \ldots, x_m)'$ is a solutions, i.e. $Ax = b$. Then,

$$a_1 x_1 + \ldots + a_m x_m = b,$$

and the last column in the extended matrix $\left[\begin{array}{cc} A & b \end{array}\right]$ is a linear combination of the columns of A. Therefore, $rank\left(\left[\begin{array}{cc} A & b \end{array}\right]\right) = rank\,(A)$.

Now, suppose that $rank\,(A) = rank\left(\left[\begin{array}{cc} A & b \end{array}\right]\right)$. Since A is an $l \times m$ matrix, its rank is less or equal to m. Since $\left[\begin{array}{cc} A & b \end{array}\right]$ is $l \times (m+1)$ and has rank at most m, its columns are linearly dependent. Thus, there is a vector $(x_1, \ldots, x_m, x_{m+1})'$ such that

$$a_1 x_1 + \ldots + a_m x_m + b x_{m+1} = 0.$$

Suppose that $x_{m+1} = 0$. In this case, b is linearly independent of the columns of A, and since the column and row ranks of a matrix are equal, $rank\left(\left[\begin{array}{cc} A & b \end{array}\right]\right) = rank\,(A) + 1$, which is a contradiction. Thus, $x_{m+1} \neq 0$ and $(-x_1/x_{m+1}, \ldots, -x_m/x_{m+1})'$ is a solution to $Ax = b$. ■

Lemma 3 *$Ax = b$ has a solution if and only if $AA^+b = b$.*

Proof. From the definition of A^+, $AA^+A = A$, and $AA^+Ax = Ax$. Suppose that x is a solution. Then, $Ax = b$, and $AA^+Ax = Ax$ implies that $AA^+b = b$. Now, suppose that $AA^+b = b$. Set $\tilde{x} = A^+b$. Then, $A\tilde{x} = AA^+b = b$, and therefore $\tilde{x}$ is a solution. ■

Next, we describe a general solution to $Ax = b$, provided that it exists.

Lemma 4 *If $Ax = b$ has a solution, then it takes the following general form:*

$$x = A^+b + \left(I_m - A^+A\right) q,$$

where q is an arbitrary m-vector.

Proof. Since there is a solution by the assumption, $AA^+b = b$. It is easy to see that, due to the property $AA^+A = A$ of a Moore-Penrose inverse, $x = A^+b + (I_m - A^+A)\, q$ is a solution:

$$\begin{aligned} A\left(A^+b + \left(I_m - A^+A\right) q\right) &= AA^+b + \left(A - AA^+A\right) q \\ &= b + (A - A)\, q \\ &= b. \end{aligned}$$

Now, suppose $\tilde{x}$ is a solution. Then, $A\tilde{x} = b = AA^+b$ or $A\left(\tilde{x} - A^+b\right) = 0$. The last equation is a homogeneous system of equations, and by Lemma 1,

$$\tilde{x} - A^+b = \left(I_m - A^+A\right) q$$

for some $q \in R^m$. ■

In the case of simultaneous equations, the identification depends on whether the structural parameters can be solved *uniquely* from the reduced-form coefficients. Here is the main identification result:

Theorem 5 *A system $Ax = b$, where A is $l \times m$, $x \in R^m$, and $b \in R^l$ has a unique solution if and only if*

$$rank\left(\left[\begin{array}{cc} A & b \end{array}\right]\right) = rank\,(A) = m.$$

Proof. Existence of a solution follows from Lemma 2, so it is left to prove uniqueness. By Lemma 4, the general solution is given by

$$x = A^{+}b + \left(I_m - A^{+}A\right)q,$$

where q is arbitrary. Thus, uniqueness of the solution is equivalent to $A^{+}A = I_m$. Suppose that $rank\,(A) = m$. In this case, A has full column rank, and $A^{+} = \left(A^{'}A\right)^{-1}A'$. It follows that $A^{+}A = \left(A^{'}A\right)^{-1}A'A = I_m$.

Suppose now that $A^{+}A = I_m$. Then, $\min\left(rank\,(A^{+})\,, rank\,(A)\right) \geq rank\,(I_m) = m$. Therefore, since A is $l \times m$, it has rank m. (The rank of A cannot exceed m. If $rank\,(A) < m$, then $rank\,(A^{+}A) < m$ and, therefore, $A^{+}A \neq I_m$.) ■

Note that in the case of simultaneous equations, the reduced-form coefficients are *defined* through the structural parameters, $\Pi = \Gamma^{-1}B$, and therefore there is at least one solution to $\Gamma\Pi = B$. Hence, identification is stated as the rank condition $rank\,(\Pi_2) = m_1$.