

Lecture 6: Effects at quantiles and quantile regression

Economics 527: Econometric Methods

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Effects at the mean and at quantiles

Effect at the mean

- One regressor:

$$Y_i = \beta_0 + \beta_1 X_i + U_i, \quad \text{E}[U_i | X_i] = 0$$

- Effect for the “average” individual, with $U_i = 0$:

$$\beta_1 = \frac{\partial(\beta_0 + \beta_1 X_i)}{\partial X_i}$$

- Under normality:

$$\begin{aligned} U_i | X_i &\sim N(0, \sigma^2) \\ \implies \text{E}[Y_i | X_i] &= \beta_0 + \beta_1 X_i \end{aligned}$$

- Average effect: β_1 .

Conditional quantile $q_\tau^Y(X_i)$

- z_τ for $Z \sim N(0, 1)$:

$$\text{P}(Z \leq z_\tau) = \tau, \quad \tau \in (0, 1)$$

- **Definition.** $q_\tau^Y(X_i)$, the τ -th quantile of Y_i given X_i :

$$\text{P}(Y_i \leq q_\tau^Y(X_i) | X_i) = \tau$$

- **Claim.**

$$q_\tau^Y(X_i) = \beta_0 + \beta_1 X_i + q_\tau^U(X_i)$$

- **Claim.** Under $U_i | X_i \sim N(0, \sigma^2)$:

$$q_\tau^U(X_i) = \sigma z_\tau$$

Verification

- Check that $\beta_0 + \beta_1 X_i + \sigma z_\tau$ is the τ -th quantile of Y_i given X_i :

$$\begin{aligned}
 & \mathbb{P}\left(Y_i \leq \beta_0 + \beta_1 X_i + \underbrace{\sigma z_\tau}_{q_\tau^U(X_i)} \mid X_i\right) \\
 &= \mathbb{P}(\beta_0 + \beta_1 X_i + U_i \leq \beta_0 + \beta_1 X_i + \sigma z_\tau \mid X_i) \\
 &= \mathbb{P}(U_i \leq \sigma z_\tau \mid X_i) \\
 &= \mathbb{P}\left(\underbrace{\frac{U_i}{\sigma}}_{N(0,1)} \leq z_\tau \mid X_i\right) \\
 &= \tau
 \end{aligned}$$

Same effect at every quantile

- Therefore:

$$\begin{aligned}
 & \boxed{q_\tau^Y(X_i) = \beta_0 + \beta_1 X_i + \sigma z_\tau} \\
 &= (\beta_0 + \sigma z_\tau) + \beta_1 X_i
 \end{aligned}$$

- Effect at the τ -th quantile:

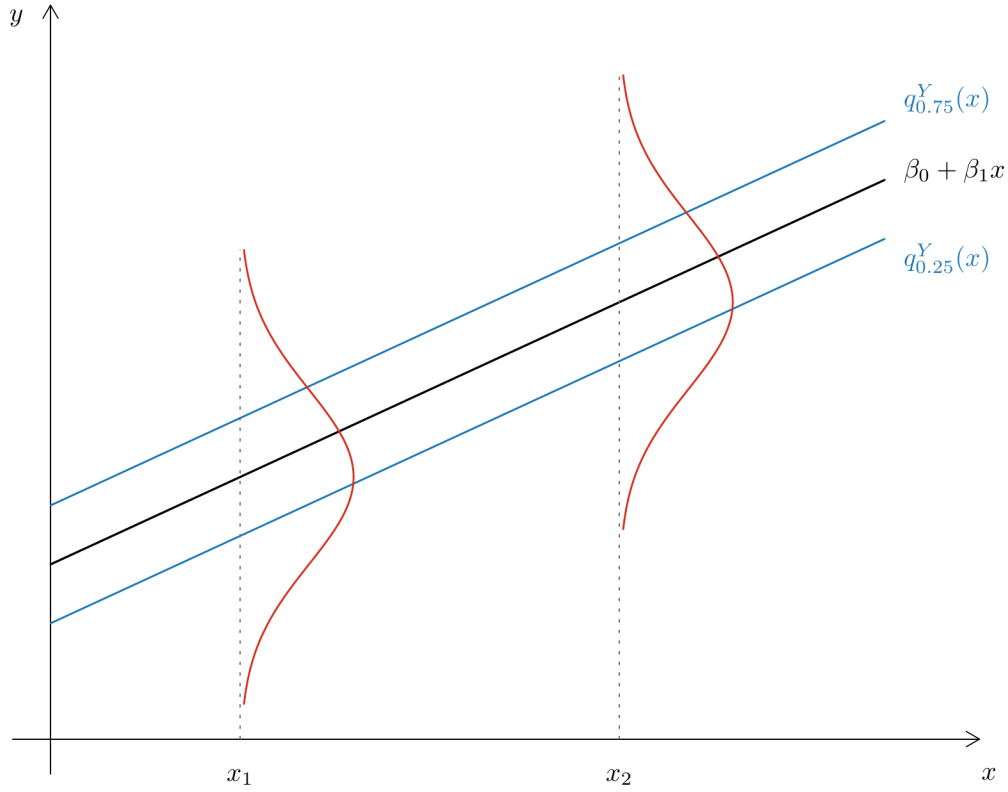
$$\begin{aligned}
 \frac{\partial q_\tau^Y(X_i)}{\partial X_i} &= \frac{\partial[\beta_0 + \beta_1 X_i + \overbrace{\sigma z_\tau}^{\text{const.}}]}{\partial X_i} \\
 &= \beta_1
 \end{aligned}$$

- Same effect β_1 at every quantile and at the mean.
- For example, $z_{0.05} \approx -1.64$:

$$q_{0.05}^Y(X_i) \approx \beta_0 + \beta_1 X_i - 1.64 \sigma$$

Equal spread and parallel quantile lines

- Densities of Y_i given $X_i = x$ on $\beta_0 + \beta_1 x$, and $q_{0.75}^Y(x)$, $q_{0.25}^Y(x)$:



Heteroskedastic normal model

- Suppose:

$$U_i \mid X_i \sim N(0, \underbrace{\sigma^2(X_i)}_{\text{not const.}})$$

$$\Rightarrow \boxed{q_\tau^Y(X_i) = \beta_0 + \beta_1 X_i + \sigma(X_i) z_\tau}$$

- Exponential scale:

$$\sigma(X_i) = e^{\gamma_0 + \gamma_1 X_i}$$

- Effect at the τ -th quantile:

$$\begin{aligned} \frac{\partial q_\tau^Y(X_i)}{\partial X_i} &= \frac{\partial [\beta_0 + \beta_1 X_i + \sigma(X_i) z_\tau]}{\partial X_i} \\ &= \beta_1 + z_\tau \frac{\partial \sigma(X_i)}{\partial X_i} \\ &= \beta_1 + z_\tau \frac{\partial e^{\gamma_0 + \gamma_1 X_i}}{\partial X_i} \\ &= \beta_1 + z_\tau \overbrace{\frac{\sigma(X_i)}{e^{\gamma_0 + \gamma_1 X_i}} \gamma_1}^{>0 \text{ if } \gamma_1 > 0} \end{aligned}$$

Effects above and below the median

- Sign of z_τ :

$$\begin{aligned} z_\tau &> 0 & \text{if } \tau > 0.5 \\ z_{0.5} &= 0 \\ z_\tau &< 0 & \text{if } \tau < 0.5 \\ z_\tau &\rightarrow -\infty & \text{as } \tau \rightarrow 0 \end{aligned}$$

- Suppose $\gamma_1 > 0$:

$$\frac{\partial q_\tau^Y(X_i)}{\partial X_i} = \beta_1 + z_\tau \sigma(X_i) \gamma_1 \begin{cases} > \beta_1 & \text{if } \tau > 0.5 \\ = \beta_1 & \text{if } \tau = 0.5 \\ < \beta_1 & \text{if } \tau < 0.5 \end{cases}$$

- At the median:

$$\begin{aligned} q_{0.5}^Y(X_i) &= \beta_0 + \beta_1 X_i \\ &= E[Y_i | X_i] \end{aligned}$$

Where heterogeneity comes from

- Given X_i , a distribution of U_i :

$$E[U_i | X_i] = 0, \quad \text{but } U_i \text{ and } X_i \text{ not independent}$$

- Scale times a standard normal:

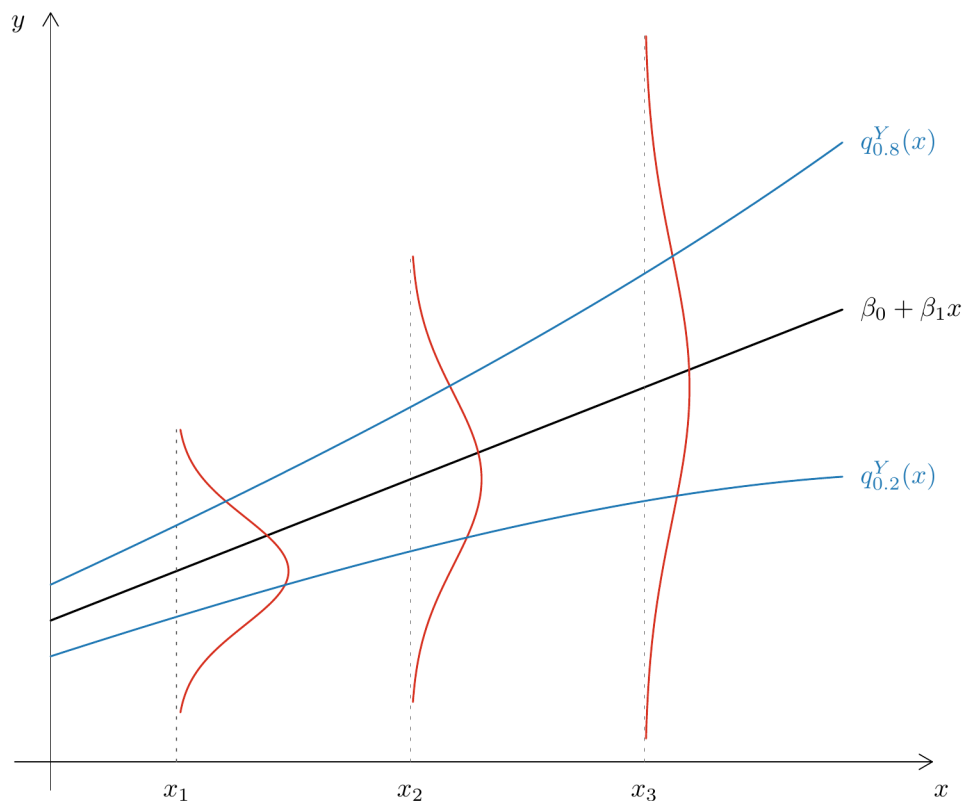
$$\begin{aligned} U_i &= \sigma(X_i) \varepsilon_i, \quad \varepsilon_i | X_i \sim N(0, 1) \\ \implies q_\tau^Y(X_i) &= \beta_0 + \beta_1 X_i + \sigma(X_i) z_\tau \end{aligned}$$

- Individual at the τ -th quantile of Y_i given X_i :

$$\begin{aligned} \varepsilon_i &= z_\tau \\ \iff Y_i &= \beta_0 + \beta_1 X_i + \sigma(X_i) z_\tau \\ &= q_\tau^Y(X_i) \end{aligned}$$

Widening densities and fanning quantile curves

- Densities on $\beta_0 + \beta_1 x$, then $q_{0.8}^Y(x)$ and $q_{0.2}^Y(x)$:



Quantile regression

Quantile regression without normality

- Explain $q_\tau^Y(X_i)$ instead of $E[Y_i | X_i]$.
- No model for $\sigma(X_i)$.
- Mean as an optimizer:

$$\underbrace{E[Y_i]}_{\mu} = \arg \min_{c \in \mathbb{R}} E \left[\begin{matrix} Y_i & c \\ 1 \times 1 & 1 \times 1 \end{matrix} \right]^2$$

- **Definition.** Check function, $\tau \in (0, 1)$:

$$\rho_\tau(u) = u(\tau - \mathbf{1}(u < 0))$$

- Two branches:

$$u < 0 \implies \rho_\tau(u) = (\tau - 1)u$$

$$u > 0 \implies \rho_\tau(u) = \tau u$$

- **Definition.** q_τ^Y , the τ -th quantile of Y_i :

$$P(Y_i \leq q_\tau^Y) = \tau$$

- Quantile as an optimizer:

$$q_\tau^Y = \arg \min_{c \in \mathbb{R}} E[\rho_\tau(Y_i - c)]$$

Median: $\tau = 0.5$

- Check function:

$$\begin{aligned}\rho_{0.5}(u) &= u(0.5 - \mathbf{1}(u < 0)) \\ &= \begin{cases} -\frac{1}{2}u & \text{if } u < 0 \\ \frac{1}{2}u & \text{if } u > 0 \end{cases} \\ &= \frac{1}{2}|u|\end{aligned}$$

- Median as an optimizer:

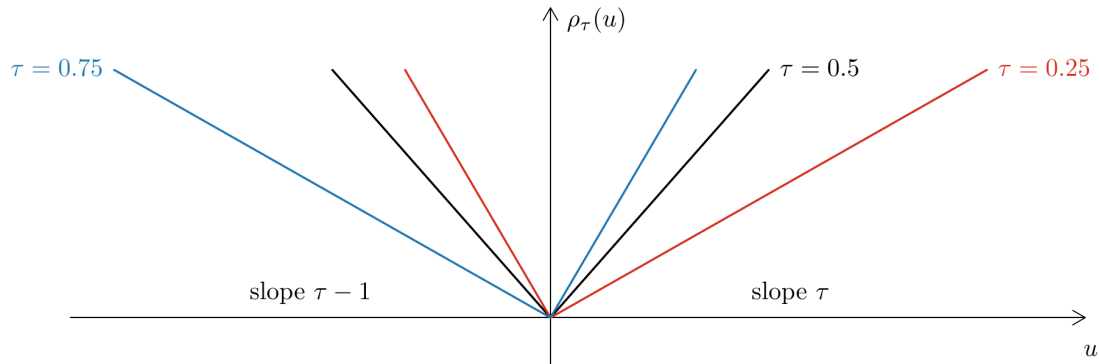
$$\begin{aligned}q_{0.5}^Y &= \arg \min_{c \in \mathbb{R}} \mathbb{E}[\rho_{0.5}(Y_i - c)] \\ &= \arg \min_{c \in \mathbb{R}} \mathbb{E}[|Y_i - c|]\end{aligned}$$

- Median:

$$\mathbb{P}(Y_i \leq q_{0.5}^Y) = 0.5$$

Why “check function”

- Graph of ρ_τ :



- $\tau < 0.5$: a check mark, $\checkmark$.

Mean regression as a minimization

- X_i now the $k \times 1$ vector of regressors, $\text{rank}(\mathbb{E}[X_i X_i^\top]) = k$; $\beta, b \in \mathbb{R}^k$.
- Population, then sample:

$$\begin{aligned}\mu &= \mathbb{E}[Y_i] = \arg \min_{c \in \mathbb{R}} \mathbb{E}[(Y_i - c)^2] \\ \hat{\mu} &= \arg \min_{c \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n (Y_i - c)^2\end{aligned}$$

- Conditional mean:

$$\mathbb{E}[Y_i | X_i] = \arg \min_{c \in \mathbb{R}} \mathbb{E}[(Y_i - c)^2 | X_i]$$

- Assume $\mathbb{E}[Y_i | X_i] = X_i^\top \beta$:

$$\beta = \arg \min_{b \in \mathbb{R}^k} \mathbb{E}[(Y_i - X_i^\top b)^2]$$

- OLS:

$$\hat{\beta} = \arg \min_{b \in \mathbb{R}^k} \frac{1}{n} \sum_{i=1}^n (Y_i - X_i^\top b)^2$$

Quantile regression by analogy

- Population, then sample:

$$q_\tau^Y = \arg \min_{c \in \mathbb{R}} \mathbb{E} [\rho_\tau(Y_i - c)]$$

$$\hat{q}_\tau^Y = \arg \min_{c \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n \rho_\tau(Y_i - c)$$

- Assume $q_\tau^Y(X_i) = X_i^\top \beta_\tau$:

$$\beta_\tau = \arg \min_{b \in \mathbb{R}^k} \mathbb{E} [\rho_\tau(Y_i - X_i^\top b)]$$

- Quantile regression:

$$\hat{\beta}_\tau = \arg \min_{b \in \mathbb{R}^k} \frac{1}{n} \sum_{i=1}^n \rho_\tau(Y_i - X_i^\top b)$$

- Estimated conditional quantile:

$$\hat{q}_\tau^Y(X_i) = X_i^\top \hat{\beta}_\tau$$

- Koenker and Bassett (1978), Econometrica.

Median regression

- Assume $q_{0.5}^Y(X_i) = X_i^\top \beta_{0.5}$:

$$\beta_{0.5} = \arg \min_{b \in \mathbb{R}^k} \mathbb{E} [\rho_{0.5}(Y_i - X_i^\top b)]$$

- Estimator, with $\rho_{0.5}(u) = \frac{1}{2} |u|$:

$$\begin{aligned} \hat{\beta}_{0.5} &= \arg \min_{b \in \mathbb{R}^k} \frac{1}{n} \sum_{i=1}^n \rho_{0.5}(Y_i - X_i^\top b) \\ &= \arg \min_{b \in \mathbb{R}^k} \frac{1}{n} \sum_{i=1}^n |Y_i - X_i^\top b| \end{aligned}$$

- Estimated conditional median:

$$\hat{q}_{0.5}^Y(X_i) = X_i^\top \hat{\beta}_{0.5}$$

Summary

Effects at quantiles

- Normal, homoskedastic: $q_\tau^Y(X_i) = \beta_0 + \beta_1 X_i + \sigma z_\tau$; $\partial q_\tau^Y(X_i) / \partial X_i = \beta_1$ at every τ , the same as at the mean.
- Normal, heteroskedastic: $q_\tau^Y(X_i) = \beta_0 + \beta_1 X_i + \sigma(X_i) z_\tau$; with $\sigma(X_i) = e^{\gamma_0 + \gamma_1 X_i}$, $\partial q_\tau^Y(X_i) / \partial X_i = \beta_1 + z_\tau \sigma(X_i) \gamma_1$.
- $\gamma_1 > 0$: effect above β_1 for $\tau > 0.5$, equal to β_1 at $\tau = 0.5$, below β_1 for $\tau < 0.5$.
- $U_i = \sigma(X_i) \varepsilon_i$, $\varepsilon_i | X_i \sim N(0, 1)$; $\varepsilon_i = z_\tau \iff Y_i = q_\tau^Y(X_i)$.

Check function and quantile regression

- $\rho_\tau(u) = u(\tau - \mathbf{1}(u < 0))$, slope $\tau - 1$ for $u < 0$ and τ for $u > 0$; $\rho_{0.5}(u) = \frac{1}{2}|u|$.
- $\mu = \arg \min_c \mathbb{E}[(Y_i - c)^2]$ and $q_\tau^Y = \arg \min_c \mathbb{E}[\rho_\tau(Y_i - c)]$.
- $q_\tau^Y(X_i) = X_i^\top \beta_\tau$: $\hat{\beta}_\tau = \arg \min_b \frac{1}{n} \sum_{i=1}^n \rho_\tau(Y_i - X_i^\top b)$ and $\hat{q}_\tau^Y(X_i) = X_i^\top \hat{\beta}_\tau$; Koenker and Bassett (1978).
- $\tau = 0.5$: $\hat{\beta}_{0.5} = \arg \min_b \frac{1}{n} \sum_{i=1}^n |Y_i - X_i^\top b|$.