Lecture 4: Confidence intervals

Economics 527: Econometric Methods

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Normal estimator

Normal regression model

- **Assumption 1.** $Y = X\beta + U$ for some $\beta \in \mathbb{R}^k$.
- **Assumption 2.** $\mathrm{E}[U \mid X] = 0$.
- **Assumption 3.** For some $\sigma^2 > 0$:

$$\begin{aligned}\mathrm{Var}(U \mid X) &= \mathrm{E}\left[UU^\top \mid X\right] \\ &= \sigma^2 I_n\end{aligned}$$

- **Assumption 4.** $\mathrm{rank}(X) = k$.
- **Assumption 5.**

$$U \mid X \sim N(\underbrace{0}_{\mathrm{E}[U|X]},\ \underbrace{\sigma^2 I_n}_{\mathrm{Var}(U|X)})$$

Distribution of $\widehat{\beta}$

- We will use:

$$\begin{aligned}&\underset{p\times 1}{V} \sim N(\underset{p\times 1}{\mu},\ \underset{p\times p}{\Sigma}) \\ \Longrightarrow \quad &\underset{m\times p}{A}\,\underset{p\times 1}{V} + \underset{m\times 1}{b} \sim N(A\mu + b,\ A\Sigma A^\top)\end{aligned}$$

- Write $\widehat{\beta}$ as a linear function of U and apply the result:

$$\begin{aligned}\widehat{\beta} &= (X^\top X)^{-1}X^\top Y \\ &= \beta + \underbrace{(X^\top X)^{-1}X^\top}_{A} U \\ \Longrightarrow \quad &\boxed{\widehat{\beta} \mid X \sim N(\beta,\ \sigma^2(X^\top X)^{-1})}\end{aligned}$$

One coefficient $\widehat{\beta}_1$

- Apply the same result to the first coefficient:

$$\begin{aligned}\widehat{\beta}_1 \mid X &= \underbrace{\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}}_{e_1^\top} \widehat{\beta} \mid X \\ &\sim N\Big(e_1^\top \beta,\ \underset{1\times k}{e_1^\top}\ \overbrace{\underset{k\times k}{\sigma^2 (X^\top X)^{-1}}}^{\mathrm{Var}(\widehat{\beta}|X)}\ \underset{k\times 1}{e_1}\Big) \\ &= N\big(\beta_1,\ \mathrm{Var}\big(\widehat{\beta}_1 \mid X\big)\big)\end{aligned}$$

Partitioned regression

- Regression on $X = [\ X_1 \quad X_2\ ]$:

$$Y = \underset{1\times 1}{\beta_1}\ \underset{n\times 1}{X_1} + \underset{n\times k_2}{X_2}\ \underset{k_2\times 1}{\beta_2} + U$$

- Third lecture, the OLS coefficient on X_1, with $M_2 = I_n - X_2(X_2^\top X_2)^{-1}X_2^\top$:

$$\widehat{\beta}_1 = \frac{X_1^\top M_2 Y}{\underbrace{X_1^\top M_2 X_1}_{1\times 1}}$$

Distribution of $\widehat{\beta}_1$

- Under Assumptions 1 to 5:

$$\widehat{\beta}_1 \mid X \sim N\Big(\beta_1,\ \frac{\sigma^2}{X_1^\top M_2 X_1}\Big)$$

Generic estimator $\widehat{\theta}$

- The regression example in generic notation, conditional on X:

$$\theta = \beta_1, \qquad \widehat{\theta} = \widehat{\beta}_1, \qquad \omega^2 = \frac{\sigma^2}{X_1^\top M_2 X_1}$$

- Generic problem, an exactly normal estimator of θ:

$$\underset{1\times 1}{\widehat{\theta}} \sim N(\underset{1\times 1}{\theta},\ \underset{1\times 1}{\omega^2})$$

- ω^2: known.
- In the regression example: σ^2 known $\Longrightarrow \omega^2$ known.
- Probability that $\widehat{\theta}$ equals θ:

$$\mathrm{P}(\widehat{\theta} = \theta) = 0$$

Confidence interval

Coverage

- **Definition.** $CI_{1-\alpha}$, a confidence interval with coverage $1-\alpha$, is a random interval such that:

$$\mathrm{P}(\theta \in CI_{1-\alpha}) = 1 - \alpha$$

- Chosen in advance: $\alpha \in (0,1)$, for example 0.01, 0.05, 0.1.
- Regression example:

$$\mathrm{P}(\beta_1 \in CI_{1-\alpha} \mid X) = 1 - \alpha$$

- At least $1-\alpha$ is acceptable, less is not:

$$\mathrm{P}(\theta \in CI_{1-\alpha}) \geq 1 - \alpha$$
$$\mathrm{P}(\theta \in CI_{1-\alpha}) < 1 - \alpha \qquad \times$$

Symmetric interval

- Symmetric around $\hat\theta$, with $c > 0$:

$$CI_{1-\alpha} = [\hat\theta - c,\ \hat\theta + c]$$

- Length of the interval:

$$\begin{aligned} \text{length} &= \hat\theta + c - (\hat\theta - c) \\ &= 2c \end{aligned}$$

- Problem: choose c for coverage $1-\alpha$:

$$\begin{aligned} 1-\alpha &= \mathrm{P}(\theta \in CI_{1-\alpha}) \\ &= \mathrm{P}\Big(\underbrace{\theta}_{\text{fixed}} \in [\overbrace{\hat\theta}^{\text{random}} - \underbrace{c}_{\text{fixed}},\ \overbrace{\hat\theta}^{\text{random}} + \underbrace{c}_{\text{fixed}}]\Big) \end{aligned}$$

Standard normal quantile

- **Definition.** z_τ is the fixed number such that, for $Z \sim N(0,1)$:

$$\mathrm{P}(Z \le z_\tau) = \tau, \qquad \tau \in (0,1)$$

- The τ-th quantile of $N(0,1)$:

$$z_\tau = \Phi^{-1}(\tau), \qquad \Phi(z) = \mathrm{P}(Z \le z)$$

- For example:

$$z_{0.975} \approx 1.96, \qquad z_{0.025} \approx -1.96$$

- Symmetry around zero:

$$z_\tau = -z_{1-\tau}$$

Choice of c

- Subtract θ and divide by ω:

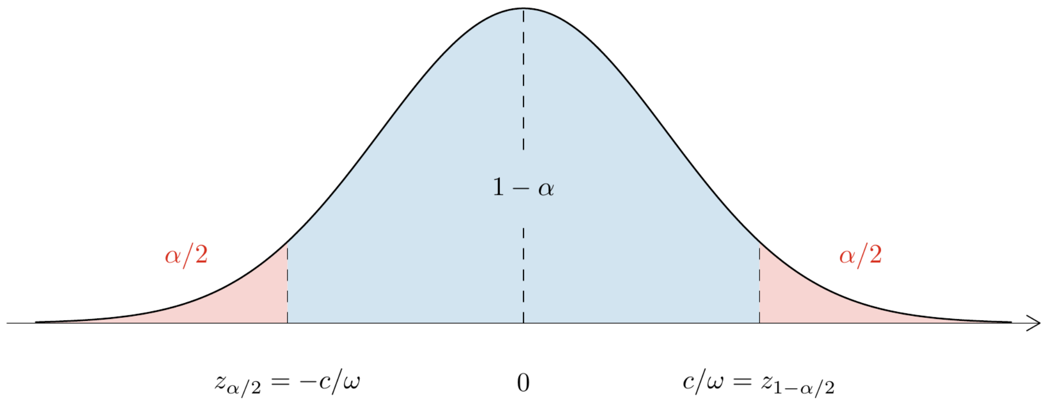
$$\begin{aligned} & \hat\theta \sim N(\theta,\ \omega^2) \\ \implies \quad & \hat\theta - \theta \sim N(0,\ \omega^2) \\ \implies \quad & \underbrace{\frac{\hat\theta - \theta}{\omega}}_{=Z} \sim N(0,\ 1) \end{aligned}$$

- Rewrite the event $\theta \in CI_{1-\alpha}$ in terms of Z:

$$\begin{aligned} \mathrm{P}(\theta \in CI_{1-\alpha}) &= \mathrm{P}(\hat\theta - c \le \theta \le \hat\theta + c) \\ &= \mathrm{P}(-c \le \theta - \hat\theta \le c) \\ &= \mathrm{P}(-c \le \hat\theta - \theta \le c) \\ &= \mathrm{P}\Big(-\frac{c}{\omega} \le \frac{\hat\theta - \theta}{\omega} \le \frac{c}{\omega}\Big) \\ &= \mathrm{P}\Big(-\frac{c}{\omega} \le Z \le \frac{c}{\omega}\Big) \end{aligned}$$

Value of c

- Standard normal density, area $1-\alpha$ between $-c/\omega$ and c/ω:



- Read c off the picture:

$$
\begin{aligned}
-\frac{c}{\omega} &= z_{\alpha/2} \\
&= -z_{1-\alpha/2} \\
\frac{c}{\omega} &= z_{1-\alpha/2} \\
\Longrightarrow \quad c &= z_{1-\alpha/2}\,\omega
\end{aligned}
$$

Coverage check

- Substitute $c = z_{1-\alpha/2}\,\omega$:

$$
\begin{aligned}
\mathrm{P}(\theta \in CI_{1-\alpha}) &= \mathrm{P}(-z_{1-\alpha/2} \le Z \le z_{1-\alpha/2}) \\
&= 1 - \frac{\alpha}{2} - \frac{\alpha}{2} \\
&= 1-\alpha
\end{aligned}
$$

where each of the two tails has probability $\alpha/2$.

Interval for θ

- Symmetric interval with coverage $1-\alpha$:

$$
\boxed{CI_{1-\alpha} = [\hat\theta - z_{1-\alpha/2}\,\omega,\ \hat\theta + z_{1-\alpha/2}\,\omega]}
$$

- Centre $\hat\theta$, half-length $\omega z_{1-\alpha/2}$.
- For $\alpha = 0.05$:

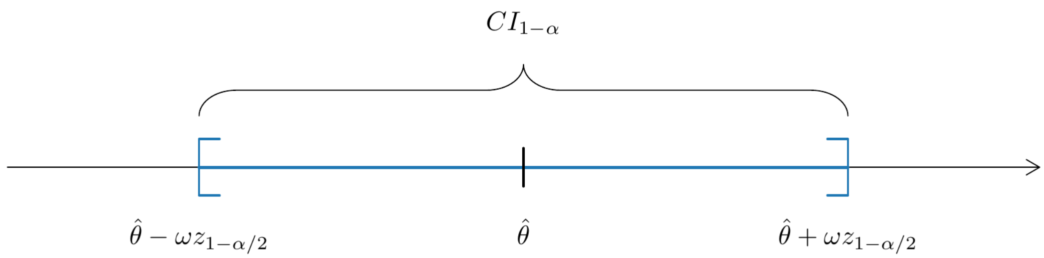
$$
z_{1-\alpha/2} = z_{0.975} \approx 1.96
$$

- Coverage, written with an absolute value:

$$
\mathrm{P}(\theta \in CI_{1-\alpha}) = \mathrm{P}\Big(\Big|\frac{\hat\theta - \theta}{\omega}\Big| \le z_{1-\alpha/2}\Big)
$$

Length of $CI_{1-\alpha}$

- $CI_{1-\alpha}$ around $\hat{\theta}$:



- Length:

$$2\,\omega\,z_{1-\alpha/2}$$

- Effect of α:

$$\alpha\downarrow \quad\Longrightarrow\quad z_{1-\alpha/2}\uparrow$$

Numerical example

- Given:

$$\begin{aligned}\hat{\theta} &= 0.06\\ \alpha = 0.05 \Longrightarrow z_{1-\alpha/2} &\approx 2.0\\ \omega &= 0.01\end{aligned}$$

- Substitute into the interval:

$$\begin{aligned}CI_{1-\alpha} &= [\hat{\theta} - z_{1-\alpha/2}\,\omega,\ \hat{\theta} + z_{1-\alpha/2}\,\omega]\\ CI_{0.95} &= [\,0.06 - 2.0\cdot 0.01,\quad 0.06 + 2.0\cdot 0.01\,]\\ &= [\,0.04,\ 0.08\,]\end{aligned}$$

- **Question.** What does $CI_{0.95} = [0.04,\ 0.08]$ mean?
  1. $\mathrm{P}(0.04 \le \theta \le 0.08) = 0.95$
  2. $\mathrm{P}(0.04 \le \hat{\theta} \le 0.08) = 0.95$
  3. $\mathrm{P}(\theta \in [0.04,\ 0.08] \mid \text{data}) = 0.95$
  4. None of the above

Random and computed interval

- Before $CI_{1-\alpha}$ is computed:

$$\mathrm{P}(\theta \in \underbrace{CI_{1-\alpha}}_{\text{random}}) = 1-\alpha$$

- Once computed:

$$CI_{0.95} = \underbrace{[\,0.04,\ 0.08\,]}_{\text{not random}}$$

Interval for β_1

- With $\hat{\theta} = \hat{\beta}_1$ and $\omega^2 = \sigma^2/(X_1^\top M_2 X_1)$:

$$CI_{1-\alpha} = \hat{\beta}_1 \pm z_{1-\alpha/2}\sqrt{\frac{\sigma^2}{X_1^\top M_2 X_1}}$$

Other intervals

Asymmetric interval

- Tails of $Z = (\hat{\theta} - \theta)/\omega$: $\alpha/3$ below, $2\alpha/3$ above.
- Interval:

$$\widetilde{CI} = [\hat{\theta} - \omega z_{1-2\alpha/3},\ \hat{\theta} + \omega z_{1-\alpha/3}]$$

- Ordering of the quantiles:

$$z_{1-2\alpha/3} < z_{1-\alpha/2} < z_{1-\alpha/3}$$

- For $\alpha = 0.1$:

$$\begin{aligned}
\alpha/2 &= 0.05 \\
1 - \alpha/2 &= 0.95 \\
1 - \alpha/3 &\approx 0.967 \\
1 - 2\alpha/3 &\approx 0.933
\end{aligned}$$

Coverage of $\widetilde{CI}$

- Coverage of $\widetilde{CI}$ is $1 - \alpha$:

$$\begin{aligned}
1 - \alpha &= 1 - \frac{2\alpha}{3} - \frac{\alpha}{3} \\
&= \left(1 - \frac{2\alpha}{3}\right) - \mathrm{P}(Z < z_{\alpha/3}) \\
&= \mathrm{P}(Z \le z_{1-2\alpha/3}) - \mathrm{P}(Z < z_{\alpha/3}) \\
&= \mathrm{P}(z_{\alpha/3} \le Z \le z_{1-2\alpha/3}) \\
&= \mathrm{P}\Big(z_{\alpha/3} \le \underbrace{\frac{\hat{\theta} - \theta}{\omega}}_{=Z} \le z_{1-2\alpha/3}\Big) \\
&= \mathrm{P}(\omega z_{\alpha/3} \le \hat{\theta} - \theta \le \omega z_{1-2\alpha/3}) \\
&= \mathrm{P}(\omega z_{\alpha/3} - \hat{\theta} \le -\theta \le \omega z_{1-2\alpha/3} - \hat{\theta}) \\
&= \mathrm{P}(\hat{\theta} - \omega z_{1-2\alpha/3} \le \theta \le \hat{\theta} - \omega z_{\alpha/3}) \\
&= \mathrm{P}(\hat{\theta} - \omega z_{1-2\alpha/3} \le \theta \le \hat{\theta} + \omega z_{1-\alpha/3}) \\
&= \mathrm{P}(\theta \in [\hat{\theta} - \omega z_{1-2\alpha/3},\ \hat{\theta} + \omega z_{1-\alpha/3}]) \\
&= \mathrm{P}(\theta \in \widetilde{CI})
\end{aligned}$$

where $z_{\alpha/3} < 0$ and $-z_{\alpha/3} = z_{1-\alpha/3}$.

Length of $\widetilde{CI}$

- Asymmetric and symmetric intervals:

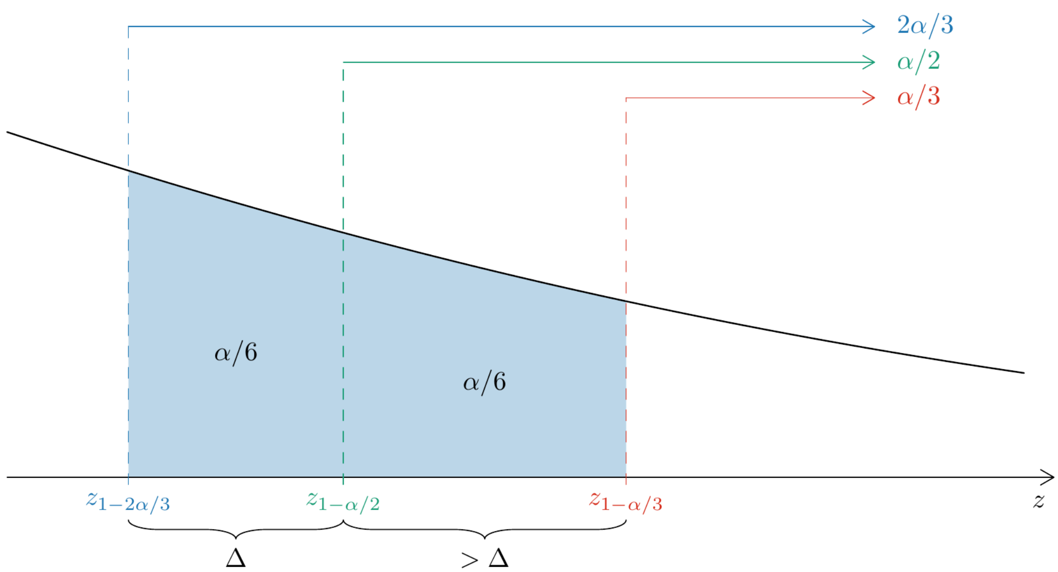
$$\begin{aligned}
\text{length of } \widetilde{CI} &= \omega(z_{1-2\alpha/3} + z_{1-\alpha/3}) \\
\text{length of } CI_{1-\alpha} &= \omega(z_{1-\alpha/2} + z_{1-\alpha/2}) \\
&= 2\omega z_{1-\alpha/2}
\end{aligned}$$

- The symmetric interval is shorter:

$$2\omega z_{1-\alpha/2} < \omega(z_{1-2\alpha/3} + z_{1-\alpha/3})$$

Upper quantiles

- Right tail of $N(0,1)$ at $\alpha = 0.1$, with three quantiles and two bands:



- Each band has the same area:

$$\begin{aligned}
\frac{2\alpha}{3} - \frac{\alpha}{2} &= \frac{\alpha}{6} \\
\frac{\alpha}{2} - \frac{\alpha}{3} &= \frac{\alpha}{6}
\end{aligned}$$

- Density of $N(0,1)$:

$$\mathrm{pdf}_{N(0,1)}(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$$

- Gaps between the quantiles:

$$\begin{aligned}
z_{1-\alpha/2} - z_{1-2\alpha/3} &= \Delta \\
z_{1-\alpha/3} - z_{1-\alpha/2} &> \Delta
\end{aligned}$$

One-sided interval

- Only the lower boundary matters:

$$[\,t,\ +\infty)$$

- Interval:

$$[\hat{\theta} - z_{1-\alpha}\,\omega,\ +\infty)$$

- Coverage:

$$\begin{aligned}
\mathrm{P}(\theta \in [\hat{\theta} - z_{1-\alpha}\,\omega,\ +\infty)) &= \mathrm{P}(\theta \geq \hat{\theta} - z_{1-\alpha}\,\omega) \\
&= \mathrm{P}\Big(z_{1-\alpha} \geq \underbrace{\frac{\hat{\theta}-\theta}{\omega}}_{=Z}\Big) \\
&= \mathrm{P}(Z \leq z_{1-\alpha}) \\
&= 1-\alpha
\end{aligned}$$

Summary

Estimator and standardization

- $\hat{\beta} \mid X \sim N(\beta, \sigma^2(X^\top X)^{-1})$ under Assumptions 1 to 5.
- $\hat{\beta}_1 \mid X \sim N(\beta_1, \sigma^2/(X_1^\top M_2 X_1))$, with $\theta = \beta_1$, $\hat{\theta} = \hat{\beta}_1$, $\omega^2 = \sigma^2/(X_1^\top M_2 X_1)$.
- $\hat{\theta} \sim N(\theta, \omega^2)$ with ω^2 known, and $\mathrm{P}(\hat{\theta} = \theta) = 0$.
- $Z = (\hat{\theta} - \theta)/\omega \sim N(0,1)$; $\mathrm{P}(Z \leq z_\tau) = \tau$ and $z_\tau = -z_{1-\tau}$.

Interval and coverage

- $\mathrm{P}(\theta \in CI_{1-\alpha}) = 1-\alpha$ with $\alpha \in (0,1)$; $CI_{1-\alpha}$ random, its computed value not random.
- $CI_{1-\alpha} = [\hat{\theta} - z_{1-\alpha/2}\,\omega,\ \hat{\theta} + z_{1-\alpha/2}\,\omega]$, of length $2\omega z_{1-\alpha/2}$.
- $\alpha = 0.05$: $z_{0.975} \approx 1.96$; $\alpha \downarrow \implies z_{1-\alpha/2} \uparrow$.
- Regression example: $CI_{1-\alpha} = \hat{\beta}_1 \pm z_{1-\alpha/2}\sqrt{\sigma^2/(X_1^\top M_2 X_1)}$, with $\mathrm{P}(\beta_1 \in CI_{1-\alpha} \mid X) = 1-\alpha$.

Asymmetric and one-sided intervals

- $\widetilde{CI} = [\hat{\theta} - \omega z_{1-2\alpha/3},\ \hat{\theta} + \omega z_{1-\alpha/3}]$: coverage $1-\alpha$.
- Length of $\widetilde{CI}$ against $CI_{1-\alpha}$: $\omega(z_{1-2\alpha/3} + z_{1-\alpha/3}) > 2\omega z_{1-\alpha/2}$.
- $[\hat{\theta} - z_{1-\alpha}\,\omega,\ +\infty)$: coverage $1-\alpha$.