LECTURE 13
SIMULTANEOUS EQUATIONS

Demand-supply system

This lecture discusses the endogeneity problem that arises from simultaneity: the left-hand-side variable and some of the right-hand-side variables are determined simultaneously. The leading example is the demand-supply system of equations:

$$Q_i^d = \gamma_1 P_i + U_{1i},$$
$$Q_i^s = \gamma_2 P_i + U_{2i},$$

where Q_i^d and Q_i^s are the quantities demanded and supplied, respectively, and P_i is the price (we can assume that $\gamma_1 < 0$ and $\gamma_2 > 0$). The system also includes the following *identity*, or equilibrium condition:

$$Q_i^d = Q_i^s = Q_i.$$

The econometrician does not observe Q_i^d and Q_i^s, but only Q_i, determined in equilibrium together with P_i. As a result, a simple regression of Q_i on P_i has no structural interpretation, since Q_i comes from both equations. Further, we can show that P_i is correlated with both U_{1i} and U_{2i}. First, we solve the system in terms of U_{1i} and U_{2i}. Subtract the demand equation from the supply equation and use the equilibrium condition to obtain:

$$P_i = \frac{U_{1i} - U_{2i}}{\gamma_2 - \gamma_1},$$

and

$$Q_i = \frac{\gamma_2 U_{1i} - \gamma_1 U_{2i}}{\gamma_2 - \gamma_1}.$$

Therefore, assuming $\mathrm{E}[U_{1i}U_{2i}] = 0$,

$$\begin{aligned}\mathrm{E}[P_i U_{1i}] &= \frac{\mathrm{E}[U_{1i}^2]}{\gamma_2 - \gamma_1} \\ &\neq 0.\end{aligned}$$

Similarly, the other three covariances $\mathrm{E}[P_i U_{2i}]$, $\mathrm{E}[Q_i U_{1i}]$, $\mathrm{E}[Q_i U_{2i}]$ are nonzero. Therefore, both Q_i and P_i are endogenous, which violates one of the critical assumptions of regression analysis. As a result, it is impossible to consistently estimate γ_1 or γ_2.

Next, assume that the demand equation includes another variable, say, income (I_i). Further, assume that I_i is excluded from the supply equation and *predetermined*, that is, it is not affected by Q_i and P_i. In fact, we assume that I_i is exogenous:

$$\mathrm{E}[I_i U_{1i}] = \mathrm{E}[I_i U_{2i}] = 0.$$

Now, the system is given by

$$Q_i^d = \gamma_1 P_i + \beta_1 I_i + U_{1i}, \tag{1}$$
$$Q_i^s = \gamma_2 P_i + U_{2i}, \tag{2}$$
$$Q_i^d = Q_i^s = Q_i. \tag{3}$$

Again, we can solve the system in terms of the predetermined variable I_i and the *shocks* U_{1i} and U_{2i}:

$$P_i = \frac{\beta_1}{\gamma_2 - \gamma_1} I_i + \frac{U_{1i} - U_{2i}}{\gamma_2 - \gamma_1},$$
$$Q_i = \frac{\gamma_2 \beta_1}{\gamma_2 - \gamma_1} I_i + \frac{\gamma_2 U_{1i} - \gamma_1 U_{2i}}{\gamma_2 - \gamma_1},$$

or

$$\begin{aligned} P_i &= \pi_1 I_i + V_{1i}, && (4)\\ Q_i &= \pi_2 I_i + V_{2i}, && (5) \end{aligned}$$

where

$$\begin{aligned} \pi_1 &= \frac{\beta_1}{\gamma_2 - \gamma_1}, \\ \pi_2 &= \frac{\gamma_2 \beta_1}{\gamma_2 - \gamma_1}, \\ V_{1i} &= \frac{U_{1i} - U_{2i}}{\gamma_2 - \gamma_1}, \\ V_{2i} &= \frac{\gamma_2 U_{1i} - \gamma_1 U_{2i}}{\gamma_2 - \gamma_1}. \end{aligned}$$

The system of equations (1)–(3) is called the *structural form*, and its parameters are referred to as the *structural coefficients*. Equations (4) and (5) are called the *reduced-form equations*, and π_1 and π_2 are the *reduced-form coefficients*. The reduced-form errors V_{1i} and V_{2i} are correlated even if the demand and supply shocks U_{1i} and U_{2i} are independent.

Since I_i is exogenous, the reduced-form equations can be consistently estimated by OLS. Economists, however, are usually interested in the structural equations. A structural equation is called identified if its coefficients can be recovered from the reduced-form parameters. In the above example,

$$\gamma_2 = \pi_2 / \pi_1. \tag{6}$$

We assume that $\pi_1 \neq 0$, or equivalently, $\beta_1 \neq 0$. Thus, the supply equation is identified, while the demand equation is not. Identification of the supply equation is possible because variation in I_i introduces exogenous shifts of the demand equation, which allows us to "see" the points on the supply line.

The parameter γ_2 can be consistently estimated by *indirect least squares* (ILS). Let $\widehat{\pi}_1$ and $\widehat{\pi}_2$ be the OLS estimators of the reduced-form coefficients π_1 and π_2, respectively. The ILS estimator of γ_2 is given by

$$\widehat{\gamma}_2^{ILS} = \frac{\widehat{\pi}_2}{\widehat{\pi}_1}.$$

The estimator $\widehat{\gamma}_2^{ILS}$ is consistent if $\widehat{\pi}_1$ and $\widehat{\pi}_2$ are consistent, as follows from (6). Further, the ILS estimator is identical to the IV estimator:

$$\begin{aligned} \widehat{\pi}_1 &= \frac{\sum_{i=1}^n I_i P_i}{\sum_{i=1}^n I_i^2}, \\ \widehat{\pi}_2 &= \frac{\sum_{i=1}^n I_i Q_i}{\sum_{i=1}^n I_i^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{\gamma}_2^{ILS} &= \frac{\sum_{i=1}^n I_i Q_i}{\sum_{i=1}^n I_i P_i} \\ &= \widehat{\gamma}_2^{IV}. \end{aligned}$$

The asymptotic distribution of the IV estimator is discussed in Lecture 10.

Consider the following system:

$$\begin{aligned} Q_i^d &= \gamma_1 P_i + \beta_1 I_i + U_{1i}, \\ Q_i^s &= \gamma_2 P_i + \beta_2 I_i + U_{2i}, \\ Q_i^d &= Q_i^s = Q_i. \end{aligned} \tag{7}$$

The reduced-form equations are the same as in (4)–(5); however, now we have

$$\pi_1 = \frac{\beta_1 - \beta_2}{\gamma_2 - \gamma_1},$$
$$\pi_2 = \frac{\gamma_2\beta_1 - \gamma_1\beta_2}{\gamma_2 - \gamma_1},$$

which cannot be solved for either of the structural coefficients. The system is not identified because changes in I_i shift both equations simultaneously. For the supply equation to be identified, I_i must be *excluded* from the supply equation ($\beta_2 = 0$).

Next, suppose that the demand equation contains two exogenous variables excluded from the supply equation:

$$\begin{aligned} Q_i^d &= \gamma_1 P_i + \beta_{11} I_i + \beta_{12} W_i + U_{1i}, \\ Q_i^s &= \gamma_2 P_i + U_{2i}, \\ Q_i^d &= Q_i^s = Q_i, \end{aligned}$$

where W_i is exogenous. The reduced form is

$$\begin{aligned} P_i &= \pi_{11} I_i + \pi_{12} W_i + V_{1i}, \\ Q_i &= \pi_{21} I_i + \pi_{22} W_i + V_{2i}, \end{aligned}$$

with

$$\begin{aligned} \pi_{11} &= \frac{\beta_{11}}{\gamma_2 - \gamma_1}, \\ \pi_{12} &= \frac{\beta_{12}}{\gamma_2 - \gamma_1}, \\ \pi_{21} &= \frac{\gamma_2\beta_{11}}{\gamma_2 - \gamma_1}, \\ \pi_{22} &= \frac{\gamma_2\beta_{12}}{\gamma_2 - \gamma_1}. \end{aligned}$$

Now, there are two solutions for γ_2:

$$\gamma_2 = \frac{\pi_{21}}{\pi_{11}} \text{ and } \gamma_2 = \frac{\pi_{22}}{\pi_{12}}.$$

As a result, the ILS produces two different estimates of γ_2. The model is overidentified, and a better approach is 2SLS. Define

$$Z_i = \begin{pmatrix} I_i \\ W_i \end{pmatrix}.$$

The 2SLS estimator of γ_2 is given by

$$\widehat{\gamma}_2^{2SLS} = \frac{\sum_{i=1}^n P_i Z_i^\top \left(\sum_{i=1}^n Z_i Z_i^\top\right)^{-1} \sum_{i=1}^n Z_i Q_i}{\sum_{i=1}^n P_i Z_i^\top \left(\sum_{i=1}^n Z_i Z_i^\top\right)^{-1} \sum_{i=1}^n Z_i P_i}.$$

If the U_{2i} are heteroskedastic (conditional on Z_i), one can use the two-step procedure to obtain the efficient GMM estimator, as discussed in Lecture 12:

$$\widehat{\gamma}_2^{GMM} = \frac{\sum_{i=1}^n P_i Z_i^\top \left(\sum_{i=1}^n \widehat{U}_{2i}^2 Z_i Z_i^\top\right)^{-1} \sum_{i=1}^n Z_i Q_i}{\sum_{i=1}^n P_i Z_i^\top \left(\sum_{i=1}^n \widehat{U}_{2i}^2 Z_i Z_i^\top\right)^{-1} \sum_{i=1}^n Z_i P_i},$$

where $\widehat{U}_{2i} = Q_i - \widehat{\gamma}_2^{2SLS} P_i$.

Identification and estimation

We now formalize the demand-supply example by introducing a general framework. Consider a system of m simultaneous equations. Let $y_{1i}, \ldots, y_{mi}$ be the m random endogenous variables, and $Z_i = (z_{1i}, \ldots, z_{li})^\top$ be the random l-vector of exogenous variables. The variable y_{ji} appears on the left-hand side of equation j, $j = 1, \ldots, m$. Let Y_{ji} denote the m_j-vector of the right-hand-side endogenous variables included in the j-th equation. Similarly, the random l_j-vector Z_{ji} denotes the right-hand-side exogenous variables included in equation j. Write u_{ji} for the random shock to equation j. The j-th equation is then

$$y_{ji} = Y_{ji}^\top \gamma_j + Z_{ji}^\top \beta_j + u_{ji},$$

where $\mathrm{E}\left[Z_i u_{ji}\right] = 0$, $\gamma_j \in \mathbb{R}^{m_j}$, and $\beta_j \in \mathbb{R}^{l_j}$. Each equation describes an IV regression model. Define

$$X_{ji} = \begin{pmatrix} Y_{ji} \\ Z_{ji} \end{pmatrix},$$
$$\delta_j = \begin{pmatrix} \gamma_j \\ \beta_j \end{pmatrix}.$$

Then, the above equation can be written as

$$y_{ji} = X_{ji}^\top \delta_j + u_{ji},$$

where we know that m_j out of $m_j + l_j$ regressors are endogenous. There are l instrumental variables in Z_i available for estimation. The moment condition for equation j is given by

$$\begin{aligned} 0 &= \mathrm{E}\left[Z_i \left(y_{ji} - X_{ji}^\top \delta_j\right)\right] \\ &= \mathrm{E}\left[Z_i y_{ji} - Z_i X_{ji}^\top \delta_j\right]. \end{aligned} \tag{8}$$

The GMM estimation requires that the $l \times (m_j + l_j)$ matrix $\mathrm{E}[Z_i X_{ji}^\top]$ has full column rank $m_j + l_j$ (the rank condition). Thus, the (necessary) order condition for identification of equation j is $l \geq m_j + l_j$, or

$$l - l_j \geq m_j.$$

The order condition says that for equation j to be identified, the number of *exogenous* regressors *excluded* from that equation must be at least as large as the number of *included endogenous* regressors. Given that equation j is identified, we can proceed to estimation. If the order and rank conditions are satisfied, one can estimate δ_j by GMM as

$$\widetilde{\delta}_j^{GMM} = \left(\sum_{i=1}^n X_{ji} Z_i^\top \left(A_{jn}^\top A_{jn}\right) \sum_{i=1}^n Z_i X_{ji}^\top \right)^{-1} \sum_{i=1}^n X_{ji} Z_i^\top \left(A_{jn}^\top A_{jn}\right) \sum_{i=1}^n Z_i y_{ji}.$$

For the efficient single-equation GMM estimator,

$$A_{jn}^\top A_{jn} \to_p \left(\mathrm{E}[u_{ji}^2 Z_i Z_i^\top]\right)^{-1}.$$

In the case of homoskedastic errors, that is, when

$$\mathrm{E}\left[u_{ji}^2 \mid Z_i\right] = \sigma_{jj} \text{ for all } i, \tag{9}$$

we need that

$$A_{jn}^\top A_{jn} \to_p \left(\mathrm{E}[Z_i Z_i^\top]\right)^{-1}.$$

We can set

$$A_{jn}^\top A_{jn} = \left(n^{-1} \sum_{i=1}^n Z_i Z_i^\top \right)^{-1},$$

and the efficient GMM reduces to the 2SLS estimator:

$$\widetilde{\delta}_j^{2SLS} = \left(\sum_{i=1}^n X_{ji}Z_i^\top \left(\sum_{i=1}^n Z_iZ_i^\top\right)^{-1} \sum_{i=1}^n Z_iX_{ji}^\top\right)^{-1} \sum_{i=1}^n X_{ji}Z_i^\top \left(\sum_{i=1}^n Z_iZ_i^\top\right)^{-1} \sum_{i=1}^n Z_iy_{ji}.$$

The 2SLS procedure is discussed in Lecture 12. The 2SLS estimators for the identified equations are consistent and *jointly* asymptotically normal. The asymptotic variance of the j-th 2SLS estimator is

$$\sigma_{jj}\left(\mathrm{E}[X_{ji}Z_i^\top]\left(\mathrm{E}[Z_iZ_i^\top]\right)^{-1}\mathrm{E}[Z_iX_{ji}^\top]\right)^{-1}.$$

Let us assume further that, in addition to (9), the errors satisfy

$$\mathrm{E}\left[u_{ri}u_{si} \mid Z_i\right] = \sigma_{rs} \text{ for all } i. \tag{10}$$

The 2SLS estimators are *asymptotically correlated* across equations if $\sigma_{rs} \neq 0$. Assuming that equations r and s are both identified, the asymptotic covariance of $\widetilde{\delta}_r^{2SLS}$ and $\widetilde{\delta}_s^{2SLS}$ is characterized by

$$n^{1/2}\begin{pmatrix} \widetilde{\delta}_r^{2SLS} - \delta_r \\ \widetilde{\delta}_s^{2SLS} - \delta_s \end{pmatrix} \to_d N\left(0, \begin{pmatrix} \sigma_{rr}\left(Q_r^\top \Sigma_Z^{-1} Q_r\right)^{-1} & \sigma_{rs}\left(Q_r^\top \Sigma_Z^{-1} Q_r\right)^{-1} Q_r^\top \Sigma_Z^{-1} Q_s \left(Q_s^\top \Sigma_Z^{-1} Q_s\right)^{-1} \\ \cdots & \sigma_{ss}\left(Q_s^\top \Sigma_Z^{-1} Q_s\right)^{-1} \end{pmatrix}\right),$$

where

$$\begin{aligned} Q_r &= \mathrm{E}[Z_iX_{ri}^\top], \\ Q_s &= \mathrm{E}[Z_iX_{si}^\top], \\ \Sigma_Z &= \mathrm{E}[Z_iZ_i^\top]. \end{aligned}$$

The asymptotic covariances between equations are required if one wants to test restrictions on parameters across equations when $\sigma_{rs} \neq 0$.

When the errors are uncorrelated across equations conditional on Z_i, the individual 2SLS estimators are asymptotically uncorrelated and, therefore, *asymptotically independent*, since the asymptotic distribution is normal.

System estimation

So far, we have estimated each structural equation individually. When the errors are correlated across equations, however, estimating all equations jointly can exploit these cross-equation correlations to improve efficiency.

System GMM estimation can be viewed as the GLS analogue for simultaneous equations: the system estimator is efficient, while equation-by-equation 2SLS is not.

Suppose that all m equations are identified. Stacking the single-equation moment conditions from (8) yields the system moment conditions:

$$\begin{aligned} 0 &= \mathrm{E}\begin{bmatrix} Z_i\left(y_{1i} - X_{1i}^\top\delta_1\right) \\ \vdots \\ Z_i\left(y_{mi} - X_{mi}^\top\delta_m\right) \end{bmatrix} \\ &= \mathrm{E}\left[\begin{pmatrix} Z_iy_{1i} \\ \vdots \\ Z_iy_{mi} \end{pmatrix} - \begin{pmatrix} Z_iX_{1i}^\top\delta_1 \\ \vdots \\ Z_iX_{mi}^\top\delta_m \end{pmatrix}\right] \\ &= \mathrm{E}\left[\begin{pmatrix} Z_iy_{1i} \\ \vdots \\ Z_iy_{mi} \end{pmatrix} - \begin{pmatrix} Z_iX_{1i}^\top & & 0 \\ & \ddots & \\ 0 & & Z_iX_{mi}^\top \end{pmatrix}\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_m \end{pmatrix}\right]. \end{aligned} \tag{11}$$

The choice of weight matrices determines the efficiency of the system estimator, just as in the single-equation case. Let A_{jn} be an $l \times l$ matrix of full rank that assigns weights to the moment conditions of equation j. Comparing (8) and (11), we deduce that the system GMM estimator is given by

$$
\begin{aligned}
&\begin{pmatrix} \widehat{\delta}_1^{GMM} \\ \vdots \\ \widehat{\delta}_m^{GMM} \end{pmatrix} \\
&= \left(\begin{pmatrix} \sum_{i=1}^n X_{1i} Z_i^\top & & 0 \\ & \ddots & \\ 0 & & \sum_{i=1}^n X_{mi} Z_i^\top \end{pmatrix} \begin{pmatrix} A_{1n}^\top A_{1n} & \dots & A_{1n}^\top A_{mn} \\ \dots & \dots & \dots \\ A_{mn}^\top A_{1n} & \dots & A_{mn}^\top A_{mn} \end{pmatrix} \right. \\
&\qquad \left. \times \begin{pmatrix} \sum_{i=1}^n Z_i X_{1i}^\top & & 0 \\ & \ddots & \\ 0 & & \sum_{i=1}^n Z_i X_{mi}^\top \end{pmatrix} \right)^{-1} \\
&\quad \times \begin{pmatrix} \sum_{i=1}^n X_{1i} Z_i^\top & & 0 \\ & \ddots & \\ 0 & & \sum_{i=1}^n X_{mi} Z_i^\top \end{pmatrix} \begin{pmatrix} A_{1n}^\top A_{1n} & \dots & A_{1n}^\top A_{mn} \\ \dots & \dots & \dots \\ A_{mn}^\top A_{1n} & \dots & A_{mn}^\top A_{mn} \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \sum_{i=1}^n Z_i y_{1i} \\ \vdots \\ \sum_{i=1}^n Z_i y_{mi} \end{pmatrix}.
\end{aligned}
$$

It remains to specify the weight matrices. Under the homoskedasticity conditions (9) and (10), the optimal weight matrices satisfy

$$
\begin{pmatrix} A_{1n}^\top A_{1n} & \dots & A_{1n}^\top A_{mn} \\ \dots & \dots & \dots \\ A_{mn}^\top A_{1n} & \dots & A_{mn}^\top A_{mn} \end{pmatrix} \to_p \begin{pmatrix} \sigma_{11} \operatorname{E}[Z_i Z_i^\top] & \dots & \sigma_{1m} \operatorname{E}[Z_i Z_i^\top] \\ \dots & \dots & \dots \\ \sigma_{m1} \operatorname{E}[Z_i Z_i^\top] & \dots & \sigma_{mm} \operatorname{E}[Z_i Z_i^\top] \end{pmatrix}^{-1}.
$$

An important simplification occurs when the errors are uncorrelated across equations conditional on Z_i:

$$
\sigma_{rs} = 0 \text{ for all } r, s = 1, \dots, m, \text{ and } r \neq s.
$$

Then, the above condition for optimal weight matrices becomes

$$
\begin{pmatrix} A_{1n}^\top A_{1n} & \dots & A_{1n}^\top A_{mn} \\ \dots & \dots & \dots \\ A_{mn}^\top A_{1n} & \dots & A_{mn}^\top A_{mn} \end{pmatrix} \to_p \begin{pmatrix} \sigma_{11} \operatorname{E}[Z_i Z_i^\top] & & 0 \\ & \ddots & \\ 0 & & \sigma_{mm} \operatorname{E}[Z_i Z_i^\top] \end{pmatrix}^{-1}.
$$

In this case, one can set

$$
\begin{aligned}
A_{jn}^\top A_{jn} &= \left(n^{-1} \sum_{i=1}^n Z_i Z_i^\top \right)^{-1} \quad \text{and} \\
A_{rn}^\top A_{sn} &= 0 \text{ for } r \neq s,
\end{aligned}
$$

so the efficient system GMM estimator reduces to the individual 2SLS estimators. That is, when the errors are uncorrelated across equations, there is no efficiency gain from system estimation.