Lecture 13: Testing multiple restrictions

Economics 326 — Introduction to Econometrics II

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Multiple restrictions

- Consider the model:

$$\begin{aligned}\ln(\text{Wage}_i) = {} & \beta_0 + \beta_1 \text{Experience}_i + \beta_2 \text{Experience}_i^2 \\ & + \beta_3 \text{PrevExperience}_i + \beta_4 \text{PrevExperience}_i^2 \\ & + \beta_5 \text{Education}_i + U_i,\end{aligned}$$

  where Experience is the experience at the current job and PrevExperience is the previous experience.

- We test whether, after controlling for the experience at the current job and education, previous experience has no effect on wage:

$$H_0 : \beta_3 = 0,\ \beta_4 = 0.$$

- We have **two** restrictions on the model parameters.
- The alternative hypothesis is that at least one of the coefficients, β_3 or β_4, is different from zero:

$$H_1 : \beta_3 \neq 0 \text{ or } \beta_4 \neq 0.$$

Testing coefficients separately

- Let T_3 and T_4 be the t-statistics associated with the coefficients of PrevExperience and PrevExperience^2:

$$T_3 = \frac{\hat{\beta}_3}{\text{se}\left(\hat{\beta}_3\right)} \quad \text{and} \quad T_4 = \frac{\hat{\beta}_4}{\text{se}\left(\hat{\beta}_4\right)}.$$

- We can use T_3 and T_4 to test the significance of β_3 and β_4 **separately** using two **size** α tests:
  - Reject $H_{0,3} : \beta_3 = 0$ in favor of $H_{1,3} : \beta_3 \neq 0$ when $|T_3| > t_{n-k-1,1-\alpha/2}$.
  - Reject $H_{0,4} : \beta_4 = 0$ in favor of $H_{1,4} : \beta_4 \neq 0$ when $|T_4| > t_{n-k-1,1-\alpha/2}$.

Why combining t-tests fails

- Rejecting $H_0 : \beta_3 = 0, \beta_4 = 0$ in favor of $H_1 : \beta_3 \neq 0$ or $\beta_4 \neq 0$ when **at least** one of the two coefficients is significant at **level** α, i.e., when

$$|T_3| > t_{n-k-1,1-\alpha/2} \quad \text{or} \quad |T_4| > t_{n-k-1,1-\alpha/2},$$

  is **not** a **size** α **test**!

- If A and B are two events, then $(A \cap B) \subset A$ and therefore $P(A \cap B) \leq P(A)$.
- **When** $\beta_3 = \beta_4 = 0$:

$$\begin{aligned}
&P\big(\text{Reject } H_{0,3} \text{ or } H_{0,4}\big) \\
&= P\big(\,|T_3| > t_{n-k-1,1-\alpha/2} \textbf{ or } |T_4| > t_{n-k-1,1-\alpha/2}\big) \\
&= P\big(\,|T_3| > t_{n-k-1,1-\alpha/2}\big) \\
&\quad + P\big(\,|T_4| > t_{n-k-1,1-\alpha/2}\big) \\
&\quad - P\big(\,|T_3| > t_{n-k-1,1-\alpha/2} \textbf{ and } |T_4| > t_{n-k-1,1-\alpha/2}\big) \\
&= 2\alpha - P\big(\text{both reject}\big) \\
&\geq \alpha.
\end{aligned}$$

Testing multiple exclusion restrictions

- Consider the model

$$\begin{aligned}
Y_i = {} & \beta_0 + \beta_1 X_{1,i} + ... + \beta_q X_{q,i} \\
& + \beta_{q+1} X_{q+1,i} + ... + \beta_k X_{k,i} + U_i.
\end{aligned}$$

  We test whether the first q regressors have no effect on Y (after controlling for the other regressors).

- The null hypothesis has q **exclusion** restrictions:

$$H_0 : \beta_1 = 0,\ \beta_2 = 0,\ ...,\ \beta_q = 0.$$

- The alternative hypothesis is that at least one of the restrictions in H_0 is false:

$$H_1 : \beta_1 \neq 0 \text{ or } \beta_2 \neq 0 \text{ or } \ ... \ \text{ or } \beta_q \neq 0.$$

F-statistic

- The idea of the test is to compare the **fit** of the **unrestricted model** with that of the **null-restricted model**.
- Let SSR_{ur} denote the Residual Sum-of-Squares of the **unrestricted model**:

$$\begin{aligned}
Y_i = {} & \beta_0 + \beta_1 X_{1,i} + ... + \beta_q X_{q,i} \\
& + \beta_{q+1} X_{q+1,i} + ... + \beta_k X_{k,i} + U_i.
\end{aligned}$$

- The **restricted model** given $H_0 : \beta_1 = 0, ..., \beta_q = 0$ is

$$Y_i = \beta_0 + \beta_{q+1} X_{q+1,i} + ... + \beta_k X_{k,i} + U_i.$$

  - Let SSR_r denote the Residual Sum-of-Squares of the **restricted model**.

- The F-statistic:

$$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}.$$

  - q = **number of restrictions**;
  - $n - k - 1$ = **unrestricted residual df**, where k is the number of regressors in the **unrestricted model**.

F-statistic (intuition)

- $$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}.$$

- Since SSR can only increase when we drop regressors,

  $$SSR_r - SSR_{ur} \geq 0,$$

  and therefore $F \geq 0$.
- If the null restrictions are **true**, the excluded variables do not contribute to explaining Y (in population), so we should expect that $SSR_r - SSR_{ur}$ is small and F **is close to zero**.
- If the null restrictions are **false**, the imposed restrictions should substantially worsen the fit, so we should expect that $SSR_r - SSR_{ur}$ is large and F **is far from zero**.
- We should reject H_0 when $F > c$ where c is some positive constant.

F distribution under H_0

- $$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}.$$

- We should reject H_0 when $F > c$.
- There is a probability that $F > c$ even when H_0 is true, so we need to choose c such that $P(F > c \mid H_0 \text{ is true}) = \alpha$.
- Under H_0 and conditional on $\mathbf{X}$, the F-statistic has the F distribution with two parameters: the numerator df (q) and the denominator df ($n-k-1$):

  $$F \mid \mathbf{X} \sim F_{q,\,n-k-1}.$$

- Similarly to the standard normal and t distributions, the F distribution has been tabulated and its critical values are available in statistical tables and statistical software such as R.

F test: decision rule and p-value

- When H_0 is true, conditional on $\mathbf{X}$:

  $$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)} \sim F_{q,\,n-k-1}.$$

- Let $F_{q,n-k-1,\tau}$ be the τ-th quantile of the $F_{q,n-k-1}$ distribution.
- A size α test of $H_0 : \beta_1 = 0, \ldots, \beta_q = 0$ against $H_1 : \beta_1 \neq 0$ or ... or $\beta_q \neq 0$ is

  $$\text{Reject } H_0 \text{ when } F > F_{q,\,n-k-1,\,1-\alpha}.$$

- The p-value can be found as τ such that $F = F_{q,n-k-1,1-\tau}$. The p-value equals τ.

F distribution in R

- To compute F critical values, use `qf()`:

$$F_{q,\,n-k-1,\,1-\alpha} = \texttt{qf(1 - alpha, df1 = q, df2 = n - k - 1)}.$$

- To compute p-values from the F distribution, use `pf()`:

$$\text{p-value} = 1 - \texttt{pf(F, df1 = q, df2 = n - k - 1)}.$$

Example: data and model

- Consider the model:

$$\begin{aligned}\ln\left(\text{Wage}_i\right) = \beta_0 &+ \beta_1 \text{Experience}_i + \beta_2 \text{Experience}_i^2 \\ &+ \beta_3 \text{PrevExperience}_i + \beta_4 \text{PrevExperience}_i^2 \\ &+ \beta_5 \text{Education}_i + U_i.\end{aligned}$$

- We test

$$H_0 : \beta_3 = 0,\ \beta_4 = 0 \quad \text{against} \quad H_1 : \beta_3 \neq 0 \text{ or } \beta_4 \neq 0.$$

- $q = 2$.
- $\alpha = 0.05$.
- Data: `wage1` from the `wooldridge` R package ($n = 526$).

```
library(wooldridge)
data(wage1)
wage1$Experience <- wage1$tenure
wage1$Experience2 <- wage1$tenure^2
wage1$PrevExperience <- wage1$exper - wage1$tenure
wage1$PrevExperience2 <- (wage1$exper - wage1$tenure)^2
wage1$Education <- wage1$educ
```

Example: unrestricted model

- The unrestricted model includes all five regressors:

```
fit_ur <- lm(lwage ~ Education + Experience + Experience2
             + PrevExperience + PrevExperience2,
             data = wage1)
summary(fit_ur)
```

```

Call:
lm(formula = lwage ~ Education + Experience + Experience2 + PrevExperience +
    PrevExperience2, data = wage1)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.01561 -0.27189 -0.01607  0.27683  1.33508 
```

```
Coefficients:
                  Estimate Std. Error t value Pr(>|t|)    
(Intercept)      0.2368427  0.1028700   2.302 0.021710 *  
Education        0.0887704  0.0072131  12.307  < 2e-16 ***
Experience       0.0471914  0.0068074   6.932 1.23e-11 ***
Experience2     -0.0008518  0.0002472  -3.446 0.000615 ***
PrevExperience   0.0168997  0.0047331   3.571 0.000389 ***
PrevExperience2 -0.0003727  0.0001208  -3.086 0.002139 ** 
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4319 on 520 degrees of freedom
Multiple R-squared:  0.3461,	Adjusted R-squared:  0.3398 
F-statistic: 55.04 on 5 and 520 DF,  p-value: < 2.2e-16
```

- The residual sum-of-squares of the unrestricted model:

```r
SSR_ur <- sum(resid(fit_ur)^2)
SSR_ur
```

```
[1] 96.99788
```

- $SSR_{ur} \approx 96.998$; $n - k - 1 = 526 - 5 - 1 = 520$.

Example: restricted model

- The restricted model drops PrevExperience and PrevExperience^2:

```r
fit_r <- lm(lwage ~ Education + Experience + Experience2,
            data = wage1)
summary(fit_r)
```

```

Call:
lm(formula = lwage ~ Education + Experience + Experience2, data = wage1)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.07720 -0.28197 -0.02346  0.26859  1.41509 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.3688491  0.0908138   4.062 5.62e-05 ***
Education    0.0852822  0.0068978  12.364  < 2e-16 ***
Experience   0.0510784  0.0067937   7.518 2.43e-13 ***
Experience2 -0.0009941  0.0002463  -4.036 6.24e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4365 on 522 degrees of freedom
Multiple R-squared:  0.3294,	Adjusted R-squared:  0.3256 
F-statistic: 85.49 on 3 and 522 DF,  p-value: < 2.2e-16
```

- The residual sum-of-squares of the restricted model:

```r
SSR_r <- sum(resid(fit_r)^2)
SSR_r
```

```
[1] 99.46294
```

- $SSR_r \approx 99.463$.

Example: F-statistic

- Computing the F-statistic manually:

```
q <- 2
n <- nrow(wage1)
k <- 5
df_denom <- n - k - 1

F_stat <- ((SSR_r - SSR_ur) / q) / (SSR_ur / df_denom)
F_stat
```

```
[1] 6.607529
```

- The critical value $F_{2,520,0.95}$:

```
cv <- qf(0.95, df1 = q, df2 = df_denom)
cv
```

```
[1] 3.013057
```

- Since $F \approx 6.61 > 3.01$, at the 5% significance level we reject H_0 that previous experience has no effect on wage.
- The p-value:

```
p_val <- 1 - pf(F_stat, df1 = q, df2 = df_denom)
p_val
```

```
[1] 0.001466371
```

- We reject H_0 for any $\alpha > 0.00147$.

Example: `linearHypothesis()`

- Instead of running two models (restricted and unrestricted), we can use `linearHypothesis()` from the `car` package after estimating the **unrestricted model**.
- Testing whether previous experience has no effect ($\beta_3 = 0$, $\beta_4 = 0$):

```
library(car)
linearHypothesis(fit_ur, c("PrevExperience = 0",
                           "PrevExperience2 = 0"))
```

```

Linear hypothesis test:
PrevExperience = 0
PrevExperience2 = 0

Model 1: restricted model
Model 2: lwage ~ Education + Experience + Experience2 + PrevExperience +
    PrevExperience2

  Res.Df    RSS Df Sum of Sq      F   Pr(>F)
1    522 99.463
2    520 96.998  2    2.4651 6.6075 0.001466 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- Testing whether the experience profiles are identical ($\beta_1 = \beta_3$ and $\beta_2 = \beta_4$):

```r
linearHypothesis(fit_ur,
                 c("Experience = PrevExperience",
                   "Experience2 = PrevExperience2"))
```

```
Linear hypothesis test:
Experience - PrevExperience = 0
Experience2 - PrevExperience2 = 0

Model 1: restricted model
Model 2: lwage ~ Education + Experience + Experience2 + PrevExperience +
    PrevExperience2

  Res.Df     RSS Df Sum of Sq      F   Pr(>F)
1    522 102.297
2    520  96.998  2    5.2987 14.203 9.87e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

F and R^2

- Let R^2_{ur} denote the R^2 of the unrestricted model:

$$\begin{aligned} Y_i = &\beta_0 + \beta_1 X_{1,i} + ... + \beta_q X_{q,i} \\ &+ \beta_{q+1} X_{q+1,i} + ... + \beta_k X_{k,i} + U_i. \end{aligned}$$

- Let R^2_r denote the R^2 of the restricted model:

$$Y_i = \beta_0 + \beta_{q+1} X_{q+1,i} + ... + \beta_k X_{k,i} + U_i.$$

- The two models have the same dependent variable and therefore the same Total Sum-of-Squares:

$$SST = \sum_{i=1}^{n} (Y_i - \bar{Y})^2 = SST_{ur} = SST_r.$$

F-statistic in terms of R^2

- Since $SSR/SST = 1 - R^2$:

$$\begin{aligned} F &= \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)} \\ &= \frac{\left(\frac{SSR_r}{SST} - \frac{SSR_{ur}}{SST}\right)/q}{\frac{SSR_{ur}}{SST}/(n-k-1)} \\ &= \frac{\left((1-R^2_r) - (1-R^2_{ur})\right)/q}{(1-R^2_{ur})/(n-k-1)} \\ &= \frac{(R^2_{ur} - R^2_r)/q}{(1-R^2_{ur})/(n-k-1)}. \end{aligned}$$

- Verification with the wage example:

```r
R2_ur <- summary(fit_ur)$r.squared
R2_r  <- summary(fit_r)$r.squared
F_from_R2 <- ((R2_ur - R2_r) / q) / ((1 - R2_ur) / df_denom)
cat("F from SSR formula:", F_stat,
    "\nF from R2 formula: ", F_from_R2, "\n")
```

```
F from SSR formula: 6.607529
F from R2 formula:  6.607529
```

Testing $\beta_1 = 1$

- Suppose we want to test $H_0 : \beta_1 = 1$ against $H_1 : \beta_1 \neq 1$ in

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + U_i.$$

- The **restricted** model is

$$Y_i = \beta_0 + X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + U_i,$$

or

$$Y_i - X_{1,i} = \beta_0 + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + U_i.$$

1. Generate a new dependent variable $Y_i^* = Y_i - X_{1,i}$.
2. Regress Y^* on a constant, $X_2, \ldots, X_k$ to obtain SSR_r.
3. Estimate the unrestricted model to obtain SSR_{ur}.
4. Compute $F = \dfrac{(SSR_r - SSR_{ur})/1}{SSR_{ur}/(n-k-1)}$.

Testing $\beta_1 + \beta_2 = 1$

- Suppose we want to test $H_0 : \beta_1 + \beta_2 = 1$ against $H_1 : \beta_1 + \beta_2 \neq 1$ in

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + U_i.$$

- The **restricted** model is

$$Y_i = \beta_0 + (1 - \beta_2) X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + U_i,$$

or

$$Y_i - X_{1,i} = \beta_0 + \beta_2 (X_{2,i} - X_{1,i}) + \ldots + \beta_k X_{k,i} + U_i.$$

1. Generate a new dependent variable $Y_i^* = Y_i - X_{1,i}$.
2. Generate a new regressor $X_{2,i}^* = X_{2,i} - X_{1,i}$.
3. Regress Y^* on a constant, $X_2^*, X_3, \ldots, X_k$ to obtain SSR_r.
4. Estimate the unrestricted model to obtain SSR_{ur}.
5. Compute $F = \dfrac{(SSR_r - SSR_{ur})/1}{SSR_{ur}/(n-k-1)}$.

Relationship between F and t

- The F statistic can also be used for testing a single restriction.
- For a single restriction, the F test and the t test lead to the **same outcome** because

$$t^2_{n-k-1} = F_{1,\, n-k-1}.$$

Overall significance test

- Consider the model

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \ldots + \beta_k X_{k,i} + U_i.$$

- Suppose we want to test whether none of the regressors explain Y:

$$\begin{aligned} &H_0 : \beta_1 = \beta_2 = \ldots = \beta_k = 0 \quad (k \text{ restrictions}), \\ &H_1 : \beta_j \neq 0 \text{ for some } j = 1, \ldots, k. \end{aligned}$$

- The restricted model is $Y_i = \beta_0 + U_i$, and since $\hat{\beta}_0 = \bar{Y}$ in this model,

$$SSR_r = \sum_{i=1}^{n} (Y_i - \bar{Y})^2 = SST \quad \text{and} \quad SSR_{ur} = SSR.$$

Overall significance: F-statistic

- The F statistic for the overall significance test is

$$\begin{aligned} F &= \frac{(SSR_r - SSR_{ur})/k}{SSR_{ur}/(n-k-1)} \\ &= \frac{(SST - SSR)/k}{SSR/(n-k-1)} \\ &= \frac{SSE/k}{SSR/(n-k-1)} \\ &= \frac{R^2/k}{(1-R^2)/(n-k-1)}. \end{aligned}$$

- The F statistic for the overall significance test and its p-value are reported in the top part of R regression output:

```r
summary(fit_ur)
```

```

Call:
lm(formula = lwage ~ Education + Experience + Experience2 + PrevExperience +
    PrevExperience2, data = wage1)

Residuals:
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PrevExperience   0.0168997  0.0047331   3.571 0.000389 ***
PrevExperience2 -0.0003727  0.0001208  -3.086 0.002139 **
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```

Summary

- Individual t-tests cannot be combined to test joint hypotheses. Rejecting when **at least one** individual t-test rejects leads to a test with size greater than α.
- The F-statistic compares the fit of the unrestricted model to the restricted model:

$$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}.$$

- Under H_0, $F \mid \mathbf{X} \sim F_{q,\,n-k-1}$. Reject H_0 when $F > F_{q,\,n-k-1,\,1-\alpha}$.
- Equivalently, in terms of R^2:

$$F = \frac{(R^2_{ur} - R^2_r)/q}{(1-R^2_{ur})/(n-k-1)}.$$

- For a single restriction ($q = 1$), $F = T^2$ and the F test is equivalent to the two-sided t-test.
- The overall significance test ($H_0 : \beta_1 = ... = \beta_k = 0$) is a special case with $F = \frac{R^2/k}{(1-R^2)/(n-k-1)}$.