Lecture 9: Hypothesis testing (Part 2)

Economics 326 — Methods of Empirical Research in Economics

Vadim Marmer, UBC

The two-sided t-test

$

$

- We are testing $H_0 : \beta_1 = \beta_{1,0}$ against $H_1 : \beta_1 \neq \beta_{1,0}$.
- When σ^2 is **unknown**, we replace it with $s^2 = \frac{1}{n-2} \sum_{i=1}^n \hat{U}_i^2$.
- The t**-statistic**:

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}}\left(\hat{\beta}_1\right)}} = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\frac{s^2}{\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}}}.$$

- We also replace the standard normal critical values $z_{1-\alpha/2}$ with the t_{n-2} critical values $t_{n-2,1-\alpha/2}$.
  However, **for large** n, $t_{n-2,1-\alpha/2} \approx z_{1-\alpha/2}$.
- The two-sided t**-test**:

  Reject H_0 when $|T| > t_{n-2,1-\alpha/2}$.

The two-sided p-value

- The decision to accept or reject H_0 depends on the critical value $t_{n-2,1-\alpha/2}$.
- If $\alpha_1 > \alpha_2$ then $t_{1-\alpha_1/2} < t_{1-\alpha_2/2}$.
- Thus, it is easier to reject H_0 with the significance level α_1 since it corresponds to a smaller acceptance region.
- p-value is the **smallest** significance level α for which we can reject H_0.

The two-sided p-value

- In order to find p-value:
  1. Compute T.
  2. Find τ such that $|T| = t_{n-2,1-\tau}$.
  3. The p-value$= \tau \times 2$.
- Note that for all $\alpha > p$-value,

$$|T| = t_{n-2,1-(p\text{-value})/2} > t_{n-2,1-\alpha/2}$$

  and we will reject H_0.
- For all $\alpha \leq p$-value,

$$|T| = t_{n-2,1-(p\text{-value})/2} \leq t_{n-2,1-\alpha/2}$$

  and we will accept H_0.

Example of p-value calculation

Suppose a regression with 19 observations produced the following output:

y	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]	
x	-.6725304	.5804943	-1.16	0.263	-1.897266	.5522055
_cons	10.18197	.2509365	40.58	0.000	9.652542	10.7114

- Here, $\hat{\beta}_1 = -0.6725, \beta_{1,0} = 0$, and in the 4th column $t = -0.6725/0.5804 = -1.16$.
- Thus, $|T| = 1.16$ and df=17.
- From the t-table, the closest critical value is $t_{17,1-0.15} = 1.069$.

  (The probability that a random variable with t_{17}-distribution lies on the right of 1.16 is ≈ 0.15.)
- The p-value is then $\approx 0.15 \times 2 = 0.300$.

Stata

- We can compute critical values and p-values using Stata instead of using the tables.
- To compute standard normal critical values use:

  display invnormal(τ),

  where τ is a number between 0 and 1.
- For example: display invnormal(1-0.05/2) produces 1.959964.
- For t critical values use

  display invttail(df,τ),

  where df is the number of degrees of freedom and τ is a number between 0 and 1.

  Note that here τ is the right-tail probability!
- For example, display invttail(62,0.05/2) produces 1.9989715.

Stata

- To compute two-sided normal p-values use:

  display 2 * (1-normal (T)) .

- For example, display 2*(1-normal(1.96)) produces 0.04999579.
- To compute two-sided t-distribution p-values, use

  display 2* (ttail(df, T)) ,

  Note that ttail gives the right tail probabilities!
- For example, display 2*(ttail(62, 1.96)) produces 0.05449415.

Example

Rent	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]	
AvgInc	.011580	.0013084	8.85	0.000	.0089646	.0141954

- Stata reports the t-statistics and the p-value for $H_0 : \beta = 0$.

- To test H_0 whether the coefficient of AvgInc is zero: $T = 0.01158/0.0013084 = 8.85$.
- The p-value is extremely close to zero, (display 2*(ttail$(62, 8.85)$) gives 1.345×10^{-12}), so for all reasonable significance levels α, we reject H_0 that the coefficient of AvgInc is zero.
- AvgInc is a **statistically significant** regressor.

Example (continued)

```
      AvgInc |    .011580   .0013084     8.85   0.000     .0089646    .0141954
```

- Consider now testing H_0 that the coefficient of AvgInc is 0.009 against the alternative that it is different from 0.009.
- $T = (0.01158 - 0.009)/0.0013084 \approx 1.97$.
- At 5% significance level, $t_{62,0.975} \approx 1.999 > T$ and we accept H_0.
- At 10% significance level, $t_{62,0.95} \approx 1.67 < T$ and we reject H_0.
- The two sided p-value is 2*(ttail$(62, 1.97)$) $\Longrightarrow$ \$ \$0.053.
- For $\alpha \leq 0.053$ we will accept H_0 and for $\alpha > 0.053$ we will reject H_0.

Confidence intervals and hypothesis testing

- There is one-to-one correspondence between confidence intervals and hypothesis testing.
- We cannot reject $H_0 : \beta_1 = \beta_{1,0}$ against a two-sided alternative if $|T| \leq t_{n-2,1-\alpha/2}$ or if and only if:

$$-t_{n-2,1-\alpha/2} \leq \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}}\left(\hat{\beta}_1\right)}} \leq t_{n-2,1-\alpha/2}$$

$$\Longleftrightarrow$$

$$\hat{\beta}_1 - t_{n-2,1-\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{\beta}_1\right)} \leq \beta_{1,0} \leq \hat{\beta}_1 + t_{n-2,1-\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{\beta}_1\right)}$$

$$\Longleftrightarrow$$

$$\beta_{1,0} \in CI_{1-\alpha}.$$

- Thus, for any $\beta_{1,0} \in CI_{1-\alpha}$, we **cannot reject** $H_0 : \beta_1 = \beta_{1,0}$ against $H_1 : \beta_1 \neq \beta_{1,0}$ at significance level α.

Example

```
        Rent |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      AvgInc |    .011580   .0013084     8.85   0.000     .0089646    .0141954
```

- The 95% confidence interval for the coefficient of AvgInc is [0.0089646, 0.0141954].
- A significance level 5% test of $H_0 : \beta_1 = \beta_{1,0}$ against $H_1 : \beta_1 \neq \beta_{1,0}$ will not reject H_0 if $\beta_{1,0} \in [0.0089646, 0.0141954]$.

One-sided tests

- Consider testing $H_0 : \beta_1 \leq \beta_{1,0}$ against $H_1 : \beta_1 > \beta_{1,0}$.
- It is reasonable to reject H_0 when $\hat{\beta}_1 - \beta_{1,0}$ is large and positive or when

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}}\left(\hat{\beta}_1\right)}} > c_{1-\alpha}$$

where $c_{1-\alpha}$ is a positive constant.

- The null hypothesis H_0 is **composite**. The probability of rejection under H_0 depends on β_1.
- We pick the critical value $c_{1-\alpha}$ so that

$$P\left(\frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} > c_{1-\alpha} | \beta_1 \leq \beta_{1,0}\right) \leq \alpha$$

  for all $\beta_1 \leq \beta_{1,0}$.

One-sided tests

- For **all** $\beta_1 \leq \beta_{1,0}$,

$$\frac{\beta_1 - \beta_{1,0}}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} \leq 0,$$

  and

$$\begin{aligned}
&P\left(\frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} > c_{1-\alpha} | \beta_1 \leq \beta_{1,0}\right) \\
&= P\left(\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} + \frac{\beta_1 - \beta_{1,0}}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} > c_{1-\alpha} | \beta_1 \leq \beta_{1,0}\right) \\
&\leq P\left(\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} > c_{1-\alpha} | \beta_1 \leq \beta_{1,0}\right) \\
&= \alpha \text{ if } c_{1-\alpha} = t_{n-2,1-\alpha}.
\end{aligned}$$

One-sided tests

- For size α test, we reject $H_0 : \beta_1 \leq \beta_{1,0}$ against $H_1 : \beta_1 > \beta_{1,0}$ when

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} > t_{n-2,1-\alpha}.$$

  where $t_{n-2,1-\alpha}$ is the critical value corresponding to t-distribution with $n-2$ degrees of freedom.
  - Note that we use $1-\alpha$ and not $1-\alpha/2$ for choosing critical values in the case of one-sided testing.
- For size α test, we reject $H_0 : \beta_1 \geq \beta_{1,0}$ against $H_1 : \beta_1 < \beta_{1,0}$ when

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\mathrm{Var}}\left(\hat{\beta}_1\right)}} < -t_{n-2,1-\alpha}.$$

One-sided tests

- One-sided p-values for $H_0 : \beta_1 \leq \beta_{1,0}$ against $H_1 : \beta_1 > \beta_{1,0}$:
  1. Compute T.
  2. Find τ such that $T = t_{n-2,1-\tau}$.
  3. The p-value$= \tau$.