Lecture 3: Simple Linear Regression and OLS

Economics 326 — Introduction to Econometrics II

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Introduction

- The **simple linear regression model** is used to study the relationship between **two variables**.
- It has many limitations, but nevertheless there are examples in the literature where the simple linear regression is applied (e.g., stock returns predictability).
- It is also a good starting point to learning the regression technique.

Definitions

TABLE 2.1

Terminology for Simple Regression

y	x
Dependent variable	Independent variable
Explained variable	Explanatory variable
Response variable	Control variable
Predicted variable	Predictor variable
Regressand	Regressor

Sample and population

- The econometrician observes random data:

observation	dependent variable	regressor
1	Y_1	X_1
2	Y_2	X_2
$\vdots$	$\vdots$	$\vdots$
n	Y_n	X_n

- A pair X_i, Y_i is called an observation.
- **Sample**: $\{(X_i, Y_i) : i = 1, \dots, n\}$.
- The population is the **joint distribution** of the sample.

The model

- We model the relationship between Y and X using the **conditional expectation**:

$$E(Y_i|X_i) = \alpha + \beta X_i.$$

- **Intercept**: $\alpha = E(Y_i|X_i = 0)$.
- **Slope**: β measures the effect of a unit change in X on Y:

$$\begin{aligned}\beta &= E(Y_i|X_i = x+1) - E(Y_i|X_i = x)\\ &= [\alpha + \beta(x+1)] - [\alpha + \beta x].\end{aligned}$$

- **Marginal effect** of X on Y:

$$\beta = \frac{dE(Y_i|X_i)}{dX_i}.$$

- **The effect is the same for all** x.

The model

- α and β in $E(Y_i|X_i) = \alpha + \beta X_i$ are **unknown**.
- **Residual (error)**:

$$U_i = Y_i - E(Y_i|X_i) = Y_i - (\alpha + \beta X_i).$$

  U_i's are **unobservable**.
- **The model**:

$$\begin{aligned}Y_i &= \alpha + \beta X_i + U_i,\\ E(U_i|X_i) &= 0.\end{aligned}$$

Functional form

- We consider a model that is linear in the coefficients α, β: $Y_i = \alpha + \beta X_i + U_i$.
- The dependent variable and the regressor can be nonlinear functions of some other variables.
- The most popular function is log.

Functional form: the log-linear model

- Consider the following model:

$$\log Y_i = \alpha + \beta X_i + U_i.$$

- In this case,

$$\begin{aligned}\beta &= \frac{d\left(\log Y_i\right)}{dX_i} \\ &= \frac{dY_i/Y_i}{dX_i} = \frac{dY_i/dX_i}{Y_i}.\end{aligned}$$

- β measures **percentage** change in Y as a response to a unit change in X.
- In this model, it is assumed that the percentage change in Y is the same for all values of X (constant).
- In $\log\left(\text{Wage}_i\right) = \alpha + \beta \times \text{Education}_i + U_i$, β measures the **return** to education.

Functional form: the log-log model

- Consider the following model:

$$\log Y_i = \alpha + \beta \log X_i + U_i.$$

- In this model,

$$\begin{aligned}\beta &= \frac{d \log Y_i}{d \log X_i} \\ &= \frac{dY_i/Y_i}{dX_i/X_i} = \frac{dY_i}{dX_i}\frac{X_i}{Y_i}.\end{aligned}$$

- β measures **elasticity**: the percentage change in Y as a response to 1% change in X.
- Here, the elasticity is assumed to be the same for all values of X.
- Example: Cobb-Douglas production function:

$$Y = \alpha K^{\beta_1} L^{\beta_2} \Longrightarrow \log Y = \log \alpha + \beta_1 \log K + \beta_2 \log L$$

(two regressors: log of capital and log of labour).

Orthogonality of residuals

The model:

$$Y_i = \alpha + \beta X_i + U_i.$$

We assume that $E\left(U_i | X_i\right) = 0$.

- $EU_i = 0$.

$$EU_i \overset{\text{Law of Iterated Expectation}}{=} E\,E\left(U_i|X_i\right) = E\,0 = 0.$$

- $Cov\left(X_i, U_i\right) = EX_iU_i = 0$.

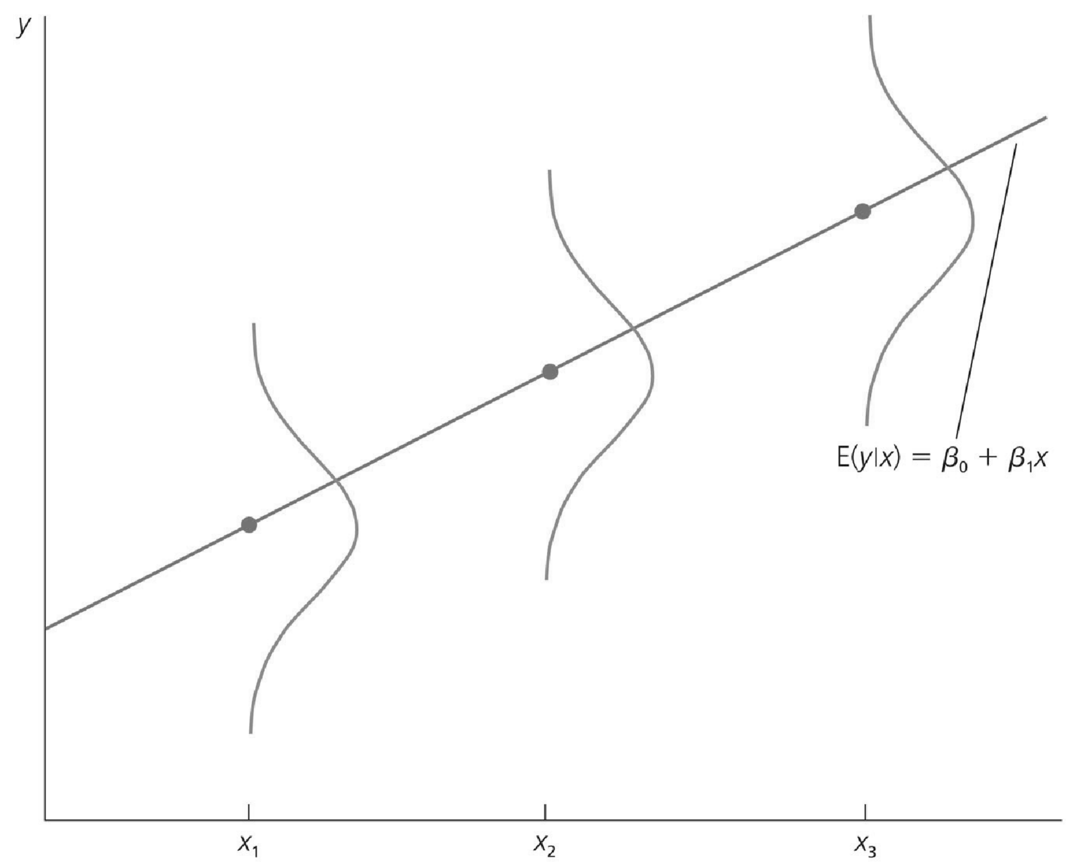
$$\begin{aligned}EX_iU_i &\overset{\text{Law of Iterated Expectation}}{=} E\,E\left(X_iU_i|X_i\right) \\ &= E\left[X_i E\left(U_i|X_i\right)\right] = E\left[X_i 0\right] = 0.\end{aligned}$$

The model

$$Y_i = \underbrace{\alpha + \beta X_i}_{\text{Predicted by } X} + \underbrace{U_i}_{\text{Orthogonal to } X}$$

FIGURE 2.1

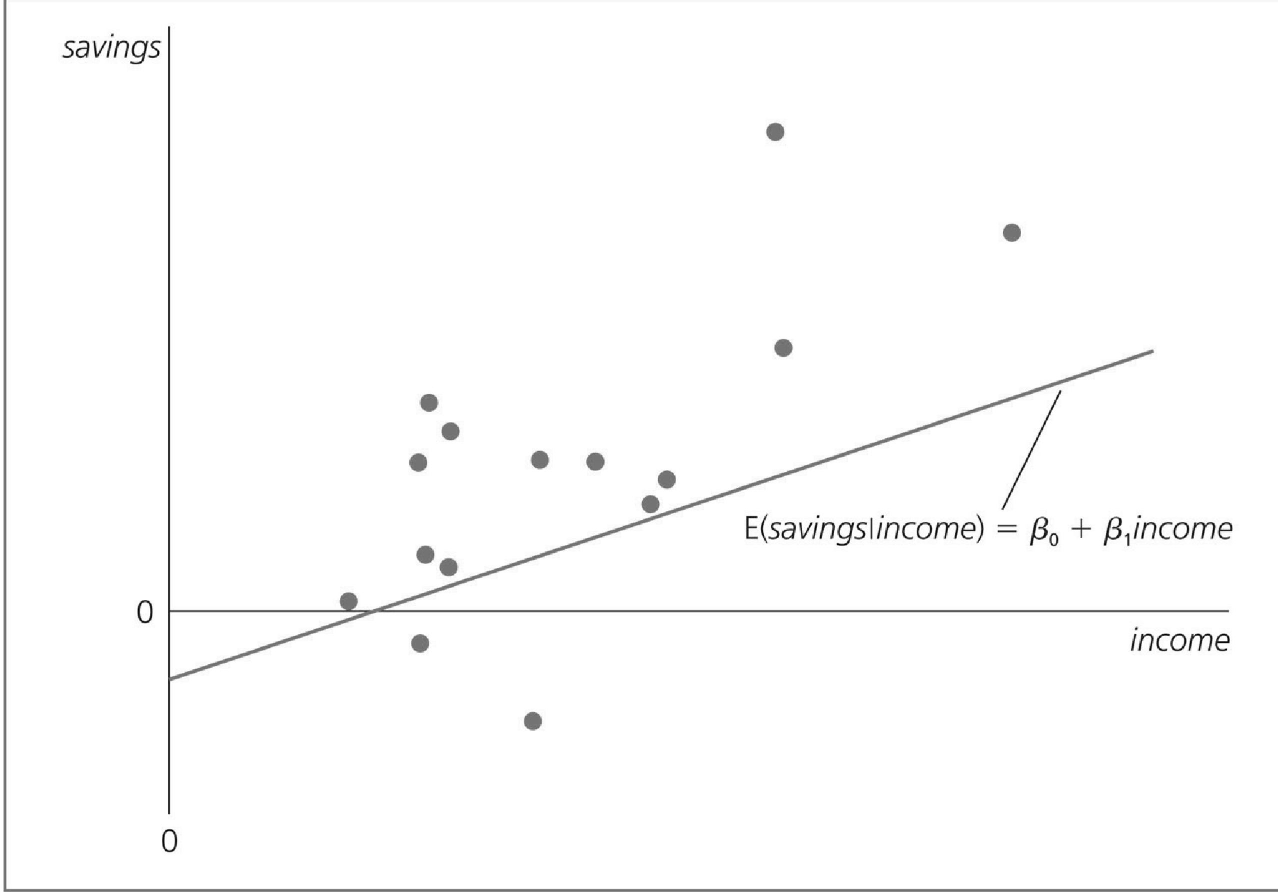
E(*y*|*x*) as a linear function of *x*.



Estimation problem

FIGURE 2.2

Scatterplot of savings and income for 15 families, and the population regression $E(savings|income) = \beta_0 + \beta_1 income$.



Problem: estimate the unknown parameters α and β using the data (n observations) on Y and X.

Method of moments

- We assume that

$$EU_i = E(Y_i - \alpha - \beta X_i) = 0.$$
$$EX_iU_i = EX_i(Y_i - \alpha - \beta X_i) = 0.$$

- An **estimator** is a function of the observable data; it can depend only on observable X and Y. Let $\hat{\alpha}$ and $\hat{\beta}$ denote the estimators of α and β.
- **Method of moments:** replace expectations with averages. **Normal equations**:

$$\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right) = 0.$$
$$\frac{1}{n}\sum_{i=1}^{n}X_i\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right) = 0.$$

Solution

- Let $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ (averages).

$$\frac{1}{n}\sum_{i=1}^{n}\left(Y_i-\hat{\alpha}-\hat{\beta}X_i\right)=0$$

implies

$$\frac{1}{n}\sum_{i=1}^{n}Y_i-\frac{1}{n}\sum_{i=1}^{n}\hat{\alpha}-\hat{\beta}\frac{1}{n}\sum_{i=1}^{n}X_i=0 \text{ or}$$
$$\bar{Y}-\hat{\alpha}-\hat{\beta}\bar{X}=0.$$

The fitted regression line goes through the averages.

- $$\hat{\alpha}=\bar{Y}-\hat{\beta}\bar{X}.$$

Solution

- $\hat{\alpha}=\bar{Y}-\hat{\beta}\bar{X}$ and therefore

$$\begin{aligned}
0&=\frac{1}{n}\sum_{i=1}^{n}X_i\left(Y_i-\hat{\alpha}-\hat{\beta}X_i\right)\\
&=\sum_{i=1}^{n}X_i\left(Y_i-\left(\bar{Y}-\hat{\beta}\bar{X}\right)-\hat{\beta}X_i\right)\\
&=\sum_{i=1}^{n}X_i\left[\left(Y_i-\bar{Y}\right)-\hat{\beta}\left(X_i-\bar{X}\right)\right]\\
&=\sum_{i=1}^{n}X_i\left(Y_i-\bar{Y}\right)-\hat{\beta}\sum_{i=1}^{n}X_i\left(X_i-\bar{X}\right).
\end{aligned}$$

Solution

- $$0=\sum_{i=1}^{n}X_i\left(Y_i-\bar{Y}\right)-\hat{\beta}\sum_{i=1}^{n}X_i\left(X_i-\bar{X}\right) \text{ or}$$
$$\hat{\beta}=\frac{\sum_{i=1}^{n}X_i\left(Y_i-\bar{Y}\right)}{\sum_{i=1}^{n}X_i\left(X_i-\bar{X}\right)}.$$

- Since

$$\sum_{i=1}^{n}X_i\left(Y_i-\bar{Y}\right)=\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)=\sum_{i=1}^{n}\left(X_i-\bar{X}\right)Y_i \text{ and}$$
$$\sum_{i=1}^{n}X_i\left(X_i-\bar{X}\right)=\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(X_i-\bar{X}\right)=\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2$$

we can also write

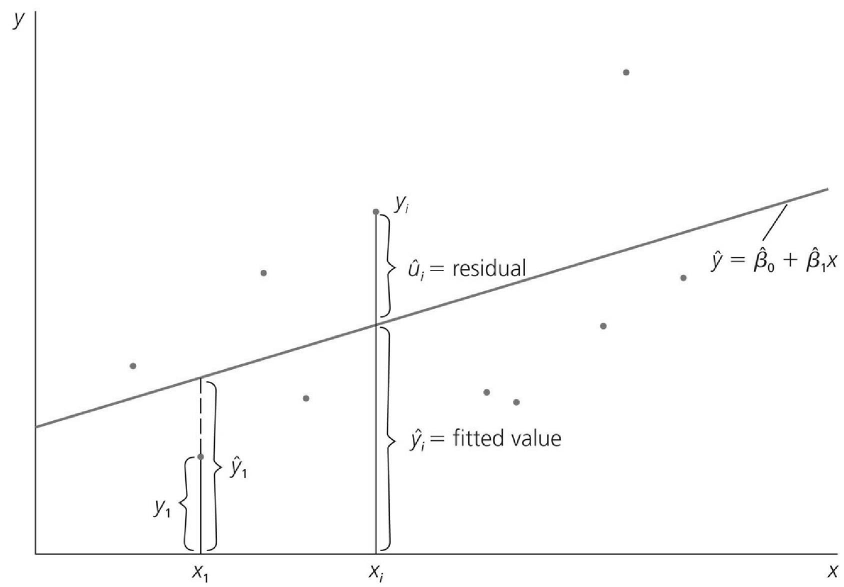
$$\hat{\beta}=\frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)Y_i}{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}.$$

Fitted line

- **Fitted values**: $\hat{Y}_i=\hat{\alpha}+\hat{\beta}X_i$.
- **Fitted residuals**: $\hat{U}_i=Y_i-\hat{\alpha}-\hat{\beta}X_i$.

FIGURE 2.4

Fitted values and residuals.

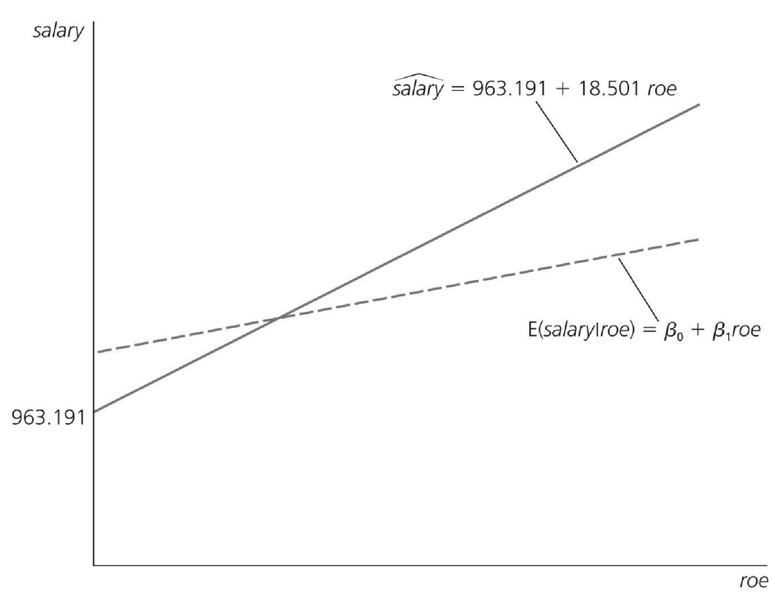


True line and fitted line

- **True:** $Y_i = \alpha + \beta X_i + U_i$, $EU_i = EX_iU_i = 0$.
- **Fitted:** $Y_i = \hat{\alpha} + \hat{\beta} X_i + \hat{U}_i$, $\sum_{i=1}^{n} \hat{U}_i = \sum_{i=1}^{n} X_i \hat{U}_i = 0$.

FIGURE 2.5

The OLS regression line $\widehat{salary}$ = 963.191 + 18.501 *roe* and the (unknown) population regression function.



Ordinary Least Squares (OLS)

- Minimize $Q(a, b) = \sum_{i=1}^{n} (Y_i - a - bX_i)^2$ with respect to a and b.

- Derivatives:

$$\frac{dQ\left(a,b\right)}{da} = -2\sum_{i=1}^{n}\left(Y_i - a - bX_i\right).$$

$$\frac{dQ\left(a,b\right)}{db} = -2\sum_{i=1}^{n}\left(Y_i - a - bX_i\right)X_i.$$

- First-order conditions:

$$0 = \sum_{i=1}^{n}\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right) = \sum_{i=1}^{n}\hat{U}_i.$$

$$0 = \sum_{i=1}^{n}\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right)X_i = \sum_{i=1}^{n}\hat{U}_iX_i.$$

- Method of moments = OLS:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)Y_i}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2} \quad \text{and} \quad \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}.$$