Lecture 2: Review of Probability

Economics 326 — Introduction to Econometrics II

Vadim Marmer, UBC

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Randomness

- **Random experiment**: an experiment the outcome of which cannot be predicted with certainty, even if the experiment is repeated under the same conditions.
- **Event**: a collection of outcomes of a random experiment.
- **Probability**: a function from events to $[0, 1]$ interval.
  - If Ω is a collection of all possible outcomes, $P(\Omega) = 1$.
  - If A is an event, $P(A) \geq 0$.
  - If $A_1, A_2, \ldots$ is a sequence of *disjoint* events, $P(A_1 \text{ or } A_2 \text{ or } \ldots) = P(A_1) + P(A_2) + \ldots$.

Random variable

- **Random variable**: a numerical representation of a random experiment.
- **Coin-flipping** example:

Outcome	X	Y	Z
Heads	0	1	-1
Tails	1	0	1

- **Rolling a dice**

Outcome	X	Y
1	1	0
2	2	1
3	3	0
4	4	1
5	5	0
6	6	1

Summation operator

- Let $\{x_i : i = 1, \ldots, n\}$ be a sequence of numbers.

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n.$$

- For a constant c:

$$\sum_{i=1}^{n} c = nc.$$

$$\sum_{i=1}^{n} cx_i = c(x_1 + x_2 + ... + x_n) = c\sum_{i=1}^{n} x_i.$$

Summation operator (continued)

- Let $\{y_i : i = 1, \dots, n\}$ be another sequence of numbers, and a, b be two constants:

$$\sum_{i=1}^{n}(ax_i + by_i) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i.$$

- But:

$$\sum_{i=1}^{n} x_i y_i \neq \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i.$$

$$\sum_{i=1}^{n} \frac{x_i}{y_i} \neq \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i}.$$

$$\sum_{i=1}^{n} x_i^2 \neq \left(\sum_{i=1}^{n} x_i\right)^2.$$

Discrete random variables

We often distinguish between **discrete** and **continuous** random variables.

- A **discrete** random variable takes on only a **finite or countably infinite** number of values.
- The **distribution** of a discrete random variable is a list of all possible values and the probability that each value would occur:

Value	x_1	x_2	...	x_n
Probability	p_1	p_2	...	p_n

Here p_i denotes the probability of a random variable X taking on value x_i:

$$p_i = P(X = x_i) \text{ (Probability Mass Function (PMF)).}$$

Each p_i is between 0 and 1, and $\sum_{i=1}^{n} p_i = 1$.

Example: Bernoulli distribution

- Consider a single trial with two outcomes: "success" (with probability p) or "failure" (with probability $1 - p$).
- Define the random variable:

$$X = \begin{cases} 1 & \text{if success} \\ 0 & \text{if failure} \end{cases}$$

- Then X follows a **Bernoulli distribution**: $X \sim \mathit{Bernoulli}(p)$.
- PMF:

$$P(X = x) = p^x(1-p)^{1-x}, \quad x \in \{0, 1\}.$$

Discrete random variables (continued)

- Indicator function:

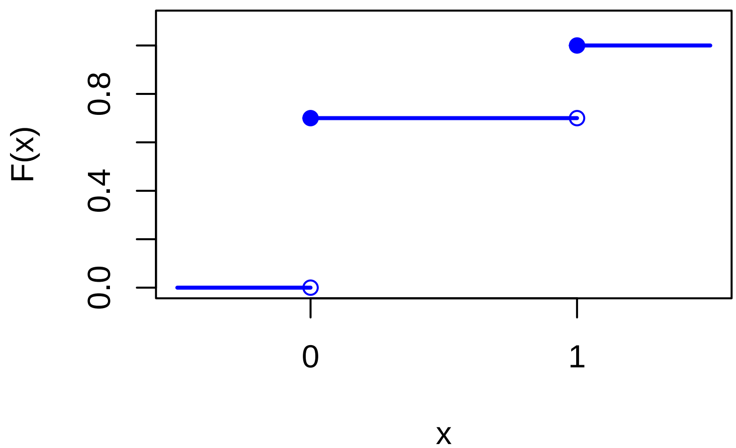
$$\mathbf{1}(x_i \le x) = \begin{cases} 1 & \text{if } x_i \le x \\ 0 & \text{if } x_i > x \end{cases}$$

- **Cumulative Distribution Function (CDF)**:

$$F(x) = P(X \le x) = \sum_i p_i \mathbf{1}(x_i \le x).$$

- $F(x)$ is non-decreasing.
- For discrete random variables, the CDF is a step function.

Example: CDF of Bernoulli(0.3)



$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Continuous random variable

- A random variable is continuously distributed if the range of possible values it can take is uncountably infinite (for example, a real line).
- A continuous random variable takes on any real value with **zero** probability.
- For continuous random variables, the CDF is continuous and differentiable.
- The derivative of the CDF is called the **Probability Density Function (PDF)**:

$$f(x) = \frac{dF(x)}{dx} \text{ and } F(x) = \int_{-\infty}^{x} f(u)du;$$

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

- Since $F(x)$ is non-decreasing, $f(x) \ge 0$ for all x.

Example: Uniform distribution

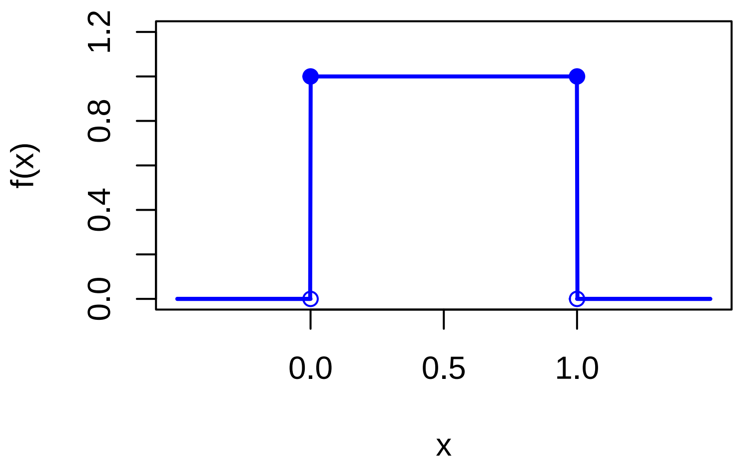
A random variable X follows a **Uniform distribution** on $[0,1]$, written $X \sim Uniform(0,1)$, if it is equally likely to take any value in $[0,1]$.

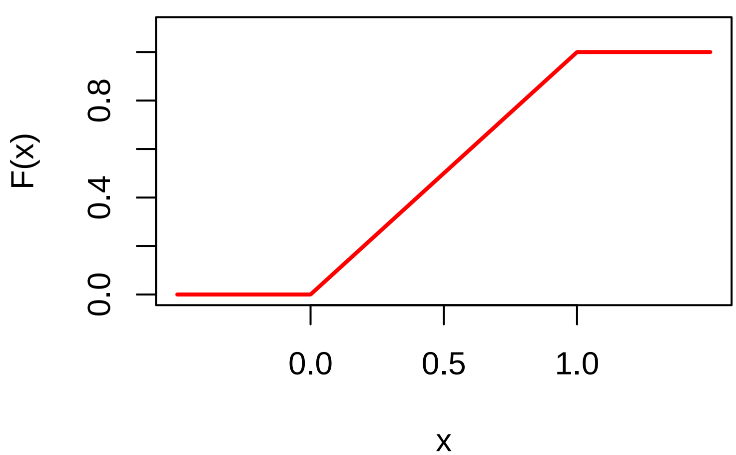
PDF:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

CDF:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$





Joint distribution (discrete)

- Two random variables X, Y

	y_1	y_2	$\cdots$	y_m	Marginal
x_1	p_{11}	p_{12}	$\cdots$	p_{1m}	$p_1^X = \sum_{j=1}^m p_{1j}$
x_2	p_{21}	p_{22}	$\cdots$	p_{2m}	$p_2^X = \sum_{j=1}^m p_{2j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	$\cdots$	p_{nm}	$p_n^X = \sum_{j=1}^m p_{nj}$

Joint PMF: $p_{ij} = P(X = x_i, Y = y_j)$.

Marginal PMF: $p_i^X = P(X = x_i) = \sum_{j=1}^{m} p_{ij}$.

- **Conditional Distribution**: If $P(X = x_1) \neq 0$,

$$p_j^{Y|X=x_1} = P(Y = y_j | X = x_1) = \frac{P(Y = y_j, X = x_1)}{P(X = x_1)} = \frac{p_{1,j}}{p_1^X}$$

Joint distribution (continuous)

- Joint PDF: $f_{X,Y}(x, y)$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$.
- Marginal PDF: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$.
- Conditional PDF: $f_{Y|X=x}(y|x) = f_{X,Y}(x, y)/f_X(x)$.

Independence

- Two (discrete) random variables are independent if **for all** x, y:

$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

- If independent:

$$P(Y = y | X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = P(Y = y).$$

- Two continuous random variables are independent if **for all** x, y:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y).$$

- If independent, $f_{Y|X}(y|x) = f_Y(y)$ for all x.

Expected value

- Let g be some function:

$$\mathrm{E}[g(X)] = \sum_i g(x_i) p_i \text{ (discrete).}$$

$$\mathrm{E}[g(X)] = \int g(x) f(x) dx \text{ (continuous).}$$

Expectation is a transformation of a distribution (PMF or PDF) and is a **constant**!

- **Mean** (center of a distribution):

$$\mathrm{E}[X] = \sum_i x_i p_i \text{ or } \mathrm{E}[X] = \int x f(x) dx.$$

- **Variance** (spread of a distribution): $\mathrm{Var}(X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$

$$\mathrm{Var}(X) = \sum_i (x_i - \mathrm{E}[X])^2 p_i \text{ or } \mathrm{Var}(X) = \int (x - \mathrm{E}[X])^2 f(x) dx.$$

- **Standard deviation**: $\sqrt{\mathrm{Var}(X)}$.

Properties

- Linearity:
  - Discrete:

$$\begin{aligned}\mathrm{E}\left[a+bX\right] &= \sum_i (a+bx_i)p_i = a\sum_i p_i + b\sum_i x_i p_i \\ &= a + b\mathrm{E}\left[X\right].\end{aligned}$$

  - Continuous:

$$\begin{aligned}\mathrm{E}\left[a+bX\right] &= \int (a+bx)f(x)dx = a\int f(x)dx + b\int xf(x)dx \\ &= a + b\mathrm{E}\left[X\right].\end{aligned}$$

- Re-centering: a random variable $X - \mathrm{E}\left[X\right]$ has mean zero:

$$\mathrm{E}\left[X - \mathrm{E}\left[X\right]\right] = \mathrm{E}\left[X\right] - \mathrm{E}\left[\mathrm{E}\left[X\right]\right] = \mathrm{E}\left[X\right] - \mathrm{E}\left[X\right] = 0.$$

Properties (continued)

- Variance formula: $\mathrm{Var}\left(X\right) = \mathrm{E}\left[X^2\right] - (\mathrm{E}\left[X\right])^2$

$$\begin{aligned}\mathrm{Var}\left(X\right) &= \mathrm{E}\left[(X - \mathrm{E}\left[X\right])^2\right] \\ &= \mathrm{E}\left[(X - \mathrm{E}\left[X\right])(X - \mathrm{E}\left[X\right])\right] \\ &= \mathrm{E}\left[(X - \mathrm{E}\left[X\right])X - (X - \mathrm{E}\left[X\right]) \cdot \mathrm{E}\left[X\right]\right] \\ &= \mathrm{E}\left[(X - \mathrm{E}\left[X\right])X\right] - \mathrm{E}\left[(X - \mathrm{E}\left[X\right]) \cdot \mathrm{E}\left[X\right]\right] \\ &= \mathrm{E}\left[X^2 - X \cdot \mathrm{E}\left[X\right]\right] - \mathrm{E}\left[X\right] \cdot \mathrm{E}\left[X - \mathrm{E}\left[X\right]\right] \\ &= \mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right] \cdot \mathrm{E}\left[X\right] - \mathrm{E}\left[X\right] \cdot 0 \\ &= \mathrm{E}\left[X^2\right] - (\mathrm{E}\left[X\right])^2\end{aligned}$$

- If $\mathrm{E}\left[X\right] = 0$ then $\mathrm{Var}\left(X\right) = \mathrm{E}\left[X^2\right]$.

Properties (continued)

- If c is a constant, $\mathrm{E}\left[c\right] = c$, and

$$\mathrm{Var}\left(c\right) = \mathrm{E}\left[(c - \mathrm{E}\left[c\right])^2\right] = (c-c)^2 = 0.$$

- $\mathrm{Var}\left(a+bX\right) = b^2\mathrm{Var}\left(X\right)$

$$\begin{aligned}\mathrm{Var}\left(a+bX\right) &= \mathrm{E}\left[((a+bX) - \mathrm{E}\left[a+bX\right])^2\right] \\ &= \mathrm{E}\left[(a+bX-a-b\mathrm{E}\left[X\right])^2\right] \\ &= \mathrm{E}\left[(bX - b\mathrm{E}\left[X\right])^2\right] = \mathrm{E}\left[b^2(X - \mathrm{E}\left[X\right])^2\right] \\ &= b^2\mathrm{E}\left[(X - \mathrm{E}\left[X\right])^2\right] \\ &= b^2\mathrm{Var}\left(X\right).\end{aligned}$$

- Re-scaling: Let $\mathrm{Var}\left(X\right) = \sigma^2$, so the standard deviation is σ:

$$\mathrm{Var}\left(\frac{X}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}\left(X\right) = 1.$$

Example: Bernoulli distribution (continued)

- Recall: $X \sim \mathit{Bernoulli}(p)$ takes values $\{0,1\}$ with $P(X=1) = p$ and $P(X=0) = 1-p$.
- Mean:

$$\mathrm{E}\left[X\right] = 0 \cdot (1-p) + 1 \cdot p = p.$$

- Variance:

$$E[X^2] = 0^2 \cdot (1-p) + 1^2 \cdot p = p.$$
$$\text{Var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p).$$

Example: Uniform distribution (continued)

- Recall: $X \sim Uniform(0,1)$ has PDF $f(x) = 1$ for $x \in [0,1]$.
- Mean:

$$E[X] = \int_0^1 x \cdot 1\, dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}.$$

- Variance:

$$E[X^2] = \int_0^1 x^2 \cdot 1\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$
$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Covariance

- **Covariance**: Let X, Y be two random variables.

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])].$$
$$\text{Cov}(X,Y) = \sum_i \sum_j (x_i - E[X])(y_j - E[Y]) \cdot P(X = x_i, Y = y_j).$$
$$\text{Cov}(X,Y) = \int\int (x - E[X])(y - E[Y]) f_{X,Y}(x,y) dx dy.$$

- $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$.

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X] \cdot E[Y].$$

Properties of covariance

- $\text{Cov}(X,c) = 0$.
- $\text{Cov}(X,X) = \text{Var}(X)$.
- $\text{Cov}(X,Y) = \text{Cov}(Y,X)$.
- $\text{Cov}(X, Y+Z) = \text{Cov}(X,Y) + \text{Cov}(X,Z)$.
- $\text{Cov}(a_1 + b_1 X, a_2 + b_2 Y) = b_1 b_2 \text{Cov}(X,Y)$.
- If X and Y are independent then $\text{Cov}(X,Y) = 0$.
- $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2\text{Cov}(X,Y)$.

Correlation

- Correlation coefficient:

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}.$$

- Cauchy-Schwartz inequality: $|\text{Cov}(X,Y)| \leq \sqrt{\text{Var}(X)\,\text{Var}(Y)}$ and therefore

$$-1 \leq \text{Corr}(X,Y) \leq 1.$$

- $\text{Corr}(X,Y) = \pm 1 \Leftrightarrow Y = a + bX$.

Conditional expectation

- Suppose you know that $X = x$. You can update your expectation of Y by **conditional expectation**:

$$\mathrm{E}[Y|X=x] = \sum_i y_i P(Y = y_i | X = x) \text{ (discrete)}$$

$$\mathrm{E}[Y|X=x] = \int y f_{Y|X}(y|x) dy \text{ (continuous).}$$

- $\mathrm{E}[Y|X=x]$ is a constant.
- $\mathrm{E}[Y|X]$ is a function of X and is a **random variable** and a function of X (Uncertainty about X has not been realized yet):

$$\mathrm{E}[Y|X] = \sum_i y_i P(Y = y_i | X) = g(X)$$

$$\mathrm{E}[Y|X] = \int y f_{Y|X}(y|X) dy = g(X),$$

for some function g that depends on PMF (PDF).

Properties of conditional expectation

- Conditional expectation satisfies all properties of unconditional expectation.
- Once you condition on X, you can treat any function of X as a constant:

$$\mathrm{E}[h_1(X) + h_2(X)Y|X] = h_1(X) + h_2(X)\mathrm{E}[Y|X],$$

for any functions h_1 and h_2.

- **Law of Iterated Expectation (LIE)**:

$$\mathrm{E}[\mathrm{E}[Y|X]] = \mathrm{E}[Y],$$

$$\mathrm{E}[\mathrm{E}[Y|X,Z]|X] = \mathrm{E}[Y|X].$$

- Conditional variance:

$$\mathrm{Var}(Y|X) = \mathrm{E}\left[(Y - \mathrm{E}[Y|X])^2 | X\right].$$

- **Mean independence**:

$$\mathrm{E}[Y|X] = \mathrm{E}[Y] = \text{constant.}$$

Example: conditional expectation and LIE

- Joint PMF of $X \in \{0,1\}$ and $Y \in \{1,2,3\}$:

	$Y=1$	$Y=2$	$Y=3$	$P(X=x)$
$X=0$	$1/10$	$2/10$	$2/10$	$1/2$
$X=1$	$2/10$	$2/10$	$1/10$	$1/2$
$P(Y=y)$	$3/10$	$4/10$	$3/10$	

- Conditional PMFs (divide joint probabilities by marginal of X):

$$P(Y=y|X=0): \quad \frac{1/10}{1/2}, \ \frac{2/10}{1/2}, \ \frac{2/10}{1/2} = \frac{1}{5}, \ \frac{2}{5}, \ \frac{2}{5}$$

$$P(Y=y|X=1): \quad \frac{2/10}{1/2}, \ \frac{2/10}{1/2}, \ \frac{1/10}{1/2} = \frac{2}{5}, \ \frac{2}{5}, \ \frac{1}{5}$$

- X and Y are not independent since the conditional PMFs of Y given $X = x$ are different for $x = 0$ and $x = 1$.
- Conditional expectations: $\mathrm{E}[Y \mid X = x] = \sum_{j=1}^{3} y_j P\left(Y = y_j \mid X = x\right)$

$$\mathrm{E}[Y|X=0] = 1 \cdot \tfrac{1}{5} + 2 \cdot \tfrac{2}{5} + 3 \cdot \tfrac{2}{5} = \tfrac{1+4+6}{5} = \tfrac{11}{5}$$

$$\mathrm{E}[Y|X=1] = 1 \cdot \tfrac{2}{5} + 2 \cdot \tfrac{2}{5} + 3 \cdot \tfrac{1}{5} = \tfrac{2+4+3}{5} = \tfrac{9}{5}$$

- LIE check: $\mathrm{E}[\mathrm{E}[Y|X]] = \tfrac{11}{5} \cdot \tfrac{1}{2} + \tfrac{9}{5} \cdot \tfrac{1}{2} = \tfrac{11}{10} + \tfrac{9}{10} = 2$
- Direct: $\mathrm{E}[Y] = 1 \cdot \tfrac{3}{10} + 2 \cdot \tfrac{4}{10} + 3 \cdot \tfrac{3}{10} = \tfrac{3+8+9}{10} = 2$

Proof of the Law of Iterated Expectations

- Let X take values $x_1, \dots, x_n$ and Y take values $y_1, \dots, y_m$.

$$\mathrm{E}[\mathrm{E}[Y|X]] = \sum_{i=1}^{n} \mathrm{E}[Y|X = x_i]\; P(X = x_i)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} y_j\, P(Y = y_j|X = x_i) \right) P(X = x_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} y_j\, P(Y = y_j|X = x_i)\, P(X = x_i)$$

$$= \sum_{j=1}^{m} y_j \sum_{i=1}^{n} P(Y = y_j,\, X = x_i)$$

$$= \sum_{j=1}^{m} y_j\, P(Y = y_j) = \mathrm{E}[Y]$$

- Key steps:
  - $P(Y = y_j|X = x_i) \cdot P(X = x_i) = P(Y = y_j,\, X = x_i)$.
  - Changing the order of summation.
  - Summing over i gives marginal $P(Y = y_j)$.

Relationship between different concepts of independence

$$\begin{array}{c} X \text{ and } Y \text{ are independent} \\ \Downarrow \\ \mathrm{E}[Y|X] = \text{constant (mean independence)} \\ \Downarrow \\ \mathrm{Cov}(X, Y) = 0 \text{ (uncorrelatedness)} \end{array}$$

Normal distribution

- A normal rv is a continuous rv that can take on any value. The PDF of a normal rv X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \text{ where}$$

$$\mu = \mathrm{E}[X] \text{ and } \sigma^2 = \mathrm{Var}(X).$$

  We usually write $X \sim N(\mu, \sigma^2)$.
- If $X \sim N(\mu, \sigma^2)$, then $a + bX \sim N(a + b\mu, b^2\sigma^2)$.

Standard Normal distribution

- Standard Normal rv has $\mu = 0$ and $\sigma^2 = 1$. Its PDF is $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$.
- Symmetric around zero (mean): if $Z \sim N(0,1)$, $P(Z > z) = P(Z < -z)$.
- Thin tails: $P(-1.96 \leq Z \leq 1.96) = 0.95$.
- If $X \sim N(\mu, \sigma^2)$, then $(X - \mu)/\sigma \sim N(0,1)$.

Bivariate Normal distribution

- X and Y have a bivariate normal distribution if their joint PDF is given by:

$$f(x,y) = \frac{1}{2\pi\sqrt{(1-\rho^2)\sigma_X^2\sigma_Y^2}} \exp\left[-\frac{Q}{2(1-\rho^2)}\right],$$

  where $Q = \frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}$,

  $\mu_X = \mathrm{E}[X], \mu_Y = \mathrm{E}[Y], \sigma_X^2 = \mathrm{Var}(X), \sigma_Y^2 = \mathrm{Var}(Y)$, and $\rho = \mathrm{Corr}(X,Y)$.

Properties of Bivariate Normal distribution

If X and Y have a bivariate normal distribution:

- $a + bX + cY \sim N(\mu^*, (\sigma^*)^2)$, where

$$\mu^* = a + b\mu_X + c\mu_Y, \quad (\sigma^*)^2 = b^2\sigma_X^2 + c^2\sigma_Y^2 + 2bc\rho\sigma_X\sigma_Y.$$

- $\mathrm{Cov}(X,Y) = 0 \Longrightarrow X$ and Y are independent.
- $\mathrm{E}[Y|X] = \mu_Y + \frac{\mathrm{Cov}(X,Y)}{\sigma_X^2}(X - \mu_X)$.
- Can be generalized to more than 2 variables (multivariate normal).

Appendix: The Cauchy-Schwartz Inequality

- Claim: $|\mathrm{Cov}(X,Y)| \leq \sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}$.
- **Proof**: Define

$$U = Y - \beta X,$$

  where

$$\beta = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)},$$

  - Note that β is a constant!
  - Also note the connection to regression and OLS in the definition of β.
- Since variances are always non-negative:

$$
\begin{aligned}
0 &\leq \operatorname{Var}(U) \\
&= \operatorname{Var}(Y - \beta X) && \text{(def. of } U\text{)} \\
&= \operatorname{Var}(Y) + \operatorname{Var}(\beta X) - 2\operatorname{Cov}(Y, \beta X) && \text{(prop. of var.)} \\
&= \operatorname{Var}(Y) + \beta^2 \operatorname{Var}(X) - 2\beta \operatorname{Cov}(X, Y) && \text{(prop. of var., cov.)} \\
&= \operatorname{Var}(Y) + \underbrace{\left(\frac{\operatorname{Cov}(X, Y)}{\operatorname{Var}(X)}\right)^2}_{=\beta^2} \operatorname{Var}(X) \\
&\qquad\qquad - 2\underbrace{\left(\frac{\operatorname{Cov}(X, Y)}{\operatorname{Var}(X)}\right)}_{=\beta} \operatorname{Cov}(X, Y) && \text{(def. of } \beta\text{)} \\
&= \operatorname{Var}(Y) + \frac{\operatorname{Cov}(X, Y)^2}{\operatorname{Var}(X)} - 2\frac{\operatorname{Cov}(X, Y)^2}{\operatorname{Var}(X)} \\
&= \operatorname{Var}(Y) - \frac{\operatorname{Cov}(X, Y)^2}{\operatorname{Var}(X)}.
\end{aligned}
$$

- Rearranging:

$$\operatorname{Cov}(X, Y)^2 \leq \operatorname{Var}(X) \operatorname{Var}(Y)$$

- or

$$|\operatorname{Cov}(X, Y)| \leq \sqrt{\operatorname{Var}(X) \operatorname{Var}(Y)}.$$